

Principal series of finite subgroups of $SU(3)$

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Abstract

We attempt to give a complete description of the “exceptional” finite subgroups $\Sigma(36 \times 3)$, $\Sigma(72 \times 3)$ and $\Sigma(216 \times 3)$ of $SU(3)$, with the aim to make them amenable to model building for fermion masses and mixing. The information on these groups which we derive contains conjugacy classes, proper normal subgroups, irreducible representations, character tables and tensor products of their three-dimensional irreducible representations. We show that, for these three exceptional groups, usage of their principal series, *i.e.* ascending chains of normal subgroups, greatly facilitates the computations and illuminates the relationship between the groups. As a preparation and testing ground for the usage of principal series, we study first the dihedral-like groups $\Delta(27)$ and $\Delta(54)$ because both are members of the principal series of the three groups discussed in the paper.

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1 Introduction

The masses of the fundamental fermions in the Standard Model are spread over more than six orders of magnitude, if we consider the charged fermions only. If we include neutrino masses, which are in the eV range or smaller, we find the additional puzzle that neutrino masses must be at least six orders of magnitude below the electron mass. Until today this mass problem is completely unsolved and no solution is anywhere near the horizon. It may well be that it is easier to understand fermion mixing than the fermion mass spectrum. There are two mixing matrices, the Cabibbo–Kobayashi–Maskawa (CKM) matrix in the quark sector and the Pontecorvo–Maki–Nakagawa–Sakata (PMNS) matrix in the lepton sector. After the discovery that the Cabibbo angle is approximately given by [1] $\sin \theta_c \simeq \sqrt{m_d/m_s}$ where m_d and m_s are the down and strange quark masses, respectively, there were many attempts to relate mixing angles and quark masses by means of horizontal symmetries—for early papers see for instance [2, 3]. In the quark sector the CKM matrix is close to the unit matrix and, therefore, relations between the small mixing angles and a hierarchical mass spectrum seems to be plausible. In the lepton sector, the situation is completely different. The “tri-bimaximal” mixing *Ansatz*

$$U_{\text{PMNS}} \simeq U_{\text{HPS}} \equiv \begin{pmatrix} 2/\sqrt{6} & 1/\sqrt{3} & 0 \\ -1/\sqrt{6} & 1/\sqrt{3} & -1/\sqrt{2} \\ -1/\sqrt{6} & 1/\sqrt{3} & 1/\sqrt{2} \end{pmatrix} \quad (1)$$

which has been put forward by Harrison, Perkins and Scott (HPS) [4] is, at present, compatible with all the experimental data [5]. In equation (1) we have left out possible Majorana phases. The simple structure of the HPS mixing matrix rather suggests that there is a symmetry responsible for lepton mixing and the mixing angles are just numbers at the (high) scale where the symmetry is effective, and corrections to U_{HPS} might be induced by renormalization-group evolution of the mixing matrix down to the electroweak scale.

Though it is by no means established that there is a symmetry origin of quark and lepton mixing, it is nevertheless worthwhile to search for possible candidates of symmetry groups. Since there are three families of fundamental fermions, it is tempting to try $SU(3)$ and its subgroups acting as family symmetries. Indeed, many models in the literature are based upon finite non-abelian subgroups of $SU(3)$ —see [6, 7] for recent reviews and [8] for a general discussion of extensions of the Standard Model. If one uses abelian symmetries, one can, at best, ensure that $\theta_{13} = 0$ in the mixing matrix while leaving the other two mixing angles as free parameters [9].

The finite subgroups of $SU(3)$ have been classified nearly 100 years ago [10]. Defining the constants

$$\omega = e^{2\pi i/3}, \quad \epsilon = e^{4\pi i/9}, \quad \beta = e^{2\pi i/7}, \quad \mu_1 = \frac{1}{2}(-1 + \sqrt{5}), \quad \mu_2 = \frac{1}{2}(-1 - \sqrt{5}), \quad (2)$$

we list the matrices

$$A(k) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & e^{2\pi i/k} & 0 \\ 0 & 0 & e^{-2\pi i/k} \end{pmatrix} \quad (k = 1, 2, 3, \dots), \quad (3a)$$

subgroup	order	generators
$\Sigma(60)$	60	A, E, W
$\Sigma(168)$	168	Y, E, Z
$\Sigma(36 \times 3)$	108	C, E, V
$\Sigma(72 \times 3)$	216	C, E, V, X
$\Sigma(216 \times 3)$	648	C, E, V, D
$\Sigma(360 \times 3)$	1080	A, E, W, F
$\Delta(3n^2) \ (n \geq 2)$	$3n^2$	$A(n), E$
$\Delta(6n^2) \ (n \geq 1)$	$6n^2$	$A(n), E, B$

Table 1: List of finite non-abelian subgroups of $SU(3)$ presented in [10, 11].

$$B = \begin{pmatrix} 0 & 0 & -1 \\ 0 & -1 & 0 \\ -1 & 0 & 0 \end{pmatrix}, \quad (3b)$$

$$D = \begin{pmatrix} \epsilon & 0 & 0 \\ 0 & \epsilon & 0 \\ 0 & 0 & \epsilon\omega \end{pmatrix}, \quad (3c)$$

$$E = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \quad (3d)$$

$$F = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & -\omega \\ 0 & -\omega^2 & 0 \end{pmatrix}, \quad (3e)$$

$$V = \frac{1}{\sqrt{3}i} \begin{pmatrix} 1 & 1 & 1 \\ 1 & \omega & \omega^2 \\ 1 & \omega^2 & \omega \end{pmatrix}, \quad (3f)$$

$$W = \frac{1}{2} \begin{pmatrix} -1 & \mu_2 & \mu_1 \\ \mu_2 & \mu_1 & -1 \\ \mu_1 & -1 & \mu_2 \end{pmatrix}, \quad (3g)$$

$$X = \frac{1}{\sqrt{3}i} \begin{pmatrix} 1 & 1 & \omega^2 \\ 1 & \omega & \omega \\ \omega & 1 & \omega \end{pmatrix}, \quad (3h)$$

$$Y = \begin{pmatrix} \beta & 0 & 0 \\ 0 & \beta^2 & 0 \\ 0 & 0 & \beta^4 \end{pmatrix}, \quad (3i)$$

$$Z = \frac{i}{\sqrt{7}} \begin{pmatrix} \beta^4 - \beta^3 & \beta^2 - \beta^5 & \beta - \beta^6 \\ \beta^2 - \beta^5 & \beta - \beta^6 & \beta^4 - \beta^3 \\ \beta - \beta^6 & \beta^4 - \beta^3 & \beta^2 - \beta^5 \end{pmatrix}, \quad (3j)$$

which occur as group generators. We use special symbols for

$$A(2) \equiv A = \text{diag}(1, -1, -1) \quad \text{and} \quad A(3) \equiv C = \text{diag}(1, \omega, \omega^2). \quad (4)$$

Then the list of finite subgroups of $SU(3)$, defined via their respective sets of generators, is given in table 1. This list consists of the two infinite series of “dihedral-like” subgroups denoted by the symbol Δ and six “exceptional” ones denoted by Σ . It was noticed that table 1 is not complete because it does not contain all possible groups of types named (C) and (D) in [10]. However, the infinitely many groups missing in table 1 are subgroups of either $\Delta(3n^2)$ (type (C)) or $\Delta(6n^2)$ (type (D)) [12, 13, 14, 15]. An example is the Frobenius group T_7 with 21 elements which is a subgroup of $\Delta(3 \times 7^2)$ —for a discussion of this group see [16].

Some of the groups in table 1 are isomorphic to well-known alternating or symmetric groups:

$$\Delta(6) \simeq S_3, \quad \Delta(12) \simeq A_4, \quad \Delta(24) \simeq S_4, \quad \Sigma(60) \simeq A_5. \quad (5)$$

Note that in the case of $\Delta(6)$ with $n = 1$, the matrix $A(1)$ is identical with the unit matrix $\mathbb{1}$ and thus not a generator.

Regarding the notation, we denoted the center of $SU(3)$ by

$$\mathbb{Z}_3 = \langle \langle \omega \mathbb{1} \rangle \rangle, \quad (6)$$

where here and in the following $G = \langle \langle M_1, \dots, M_r \rangle \rangle$ means that G is the group generated by the matrices M_1, \dots, M_r . As for the exceptional groups, we follow the notation of [11] where for $n = 36, 72, 216, 360$ the groups denoted by $\Sigma(n \times 3)$ have $3n$ elements and contain the center of $SU(3)$, whereas the groups

$$\Sigma(n) \equiv \Sigma(n \times 3) / \mathbb{Z}_3 \quad (7)$$

are *not* subgroups of $SU(3)$; their character tables and representations are discussed in [11]. The groups $\Sigma(60)$ and $\Sigma(168)$ are simple [17], *i.e.* they do not have any proper normal subgroups and, therefore, cannot comprise the center of $SU(3)$. These groups were recently used for model building in [18] and [19], respectively.

The groups S_3 , A_4 and S_4 , according to equation (5) dihedral-like groups with $n = 1$ and 2, are the most favoured finite subgroups of $SU(3)$ for model building—see for instance [6, 7, 20, 21, 22, 23] and references therein. (A very early A_4 model for quarks is found in [3].) But also the groups $\Delta(27)$ and $\Delta(54)$ with $n = 3$ have been utilized for that purpose [24, 25]. In general, the dihedral-like groups, which are discussed in detail in [12, 26, 27], have the structure

$$\Delta(3n^2) \cong (\mathbb{Z}_n \times \mathbb{Z}_n) \rtimes \mathbb{Z}_3 \quad \text{and} \quad \Delta(6n^2) \cong (\mathbb{Z}_n \times \mathbb{Z}_n) \rtimes S_3, \quad (8)$$

where the symbol “ \rtimes ” denotes the semidirect product defined in appendix A.

In this paper we employ the concept of a *principal series* (or chief series) of a group G [28] in the context of finite non-abelian subgroups of $SU(3)$. In the following we use the symbol “ \triangleleft ” where the relation $H \triangleleft G$ indicates that H is a proper normal subgroup of G . A principal series of G is defined by a series of normal subgroups

$$G_0 \equiv \{e\} \triangleleft G_1 \triangleleft \dots \triangleleft G_{\ell-1} \triangleleft G_\ell \equiv G \quad (9)$$

such that

- i. $G_j \triangleleft G_k$ holds for all $j < k$,
- ii. G_k/G_{k-1} is simple $\forall k = 1, \dots, \ell$.

The latter property states that G_{k-1} is a maximal normal subgroup of G_k . Note that G_k being a member of the principal series of G does not necessarily imply that a member of the principal series of G_k is part of the principal series of G . This will be exemplified by the principal series discussed in this paper—compare equation (11) with equations (12) and (13).

If a principal series has a reasonable length ℓ , it may be a useful concept

- to understand the structure of the group,
- to find the conjugacy classes, and
- to construct the irreducible representations (irreps) of G .

In order to elaborate on the last point we note that any irrep of a factor group G/G_k is also an irrep of G ; a sensible investigation of the irreps which exploits the principal series will start with $k = \ell$, which gives the trivial one-dimensional irrep, and go on by descending to $k = 0$, *i.e.* from the smallest to the largest group. Moreover, for all indices j, k with $0 \leq j < k \leq \ell$ the mapping

$$h_{jk} : G/G_j \rightarrow G/G_k \quad \text{with} \quad gG_j \mapsto gG_k \quad (10)$$

is a homomorphism. Therefore, all irreps of G/G_k with $k = j + 1, \dots, \ell$ are also irreps of G/G_j .

Since the dihedral-like groups are covered in detail by [12, 26, 27], we concentrate—apart from $\Delta(27)$ and $\Delta(54)$ —on the exceptional groups. As mentioned before, $\Sigma(60)$ and $\Sigma(168)$ are simple, thus their principal series are trivial. Also the principal series $\{e\} \triangleleft \mathbb{Z}_3 \triangleleft \Sigma(360 \times 3)$, derived with the help of [29], is too short for our purpose. However, the principal series

$$\{e\} \triangleleft \mathbb{Z}_3 \triangleleft \Delta(27) \triangleleft \Delta(54) \triangleleft \Sigma(36 \times 3) \triangleleft \Sigma(72 \times 3) \quad (11)$$

and

$$\{e\} \triangleleft \mathbb{Z}_3 \triangleleft \Delta(27) \triangleleft \Delta(54) \triangleleft \Sigma(72 \times 3) \triangleleft \Sigma(216 \times 3), \quad (12)$$

which will be deduced from the group generators, look promising. The aim of the paper is a discussion of $\Sigma(36 \times 3)$, $\Sigma(72 \times 3)$ and $\Sigma(216 \times 3)$ on the basis of equations (11) and (12).

The plan of the paper is as follows. Since both principal series above contain the sequence $\Delta(27) \triangleleft \Delta(54)$, we start with the principal series

$$\{e\} \triangleleft \mathbb{Z}_3 \triangleleft \mathbb{Z}_3 \times \mathbb{Z}_3 \triangleleft \Delta(27) \triangleleft \Delta(54) \quad (13)$$

in section 2; there we also derive the conjugacy classes and the character tables of $\Delta(27)$ and $\Delta(54)$, and finally we find all irreps of these two groups by taking advantage of the principal series (13). Having done this exercise, we apply our acquired knowledge

to obtain the corresponding information concerning the groups $\Sigma(36 \times 3)$, $\Sigma(72 \times 3)$ and $\Sigma(216 \times 3)$ in sections 3, 4 and 5, respectively. The conclusions are presented in section 6.

Some points are deferred to appendices: The definition and properties of semidirect products of groups, which will often be used in this paper, are expounded in appendix A. A discussion of tensor products of three-dimensional irreps of the dihedral-like groups $\Delta(27)$ and $\Delta(54)$ is presented in appendix B, the analogous but much more involved discussion of $\Sigma(36 \times 3)$ is developed in appendix C. Finally, in appendix D we investigate the nine-dimensional irreps of $\Sigma(216 \times 3)$.

In checking the computations leading to the material presented in the paper we have made extensive use of the computer algebra system GAP [29] which is very useful for the investigation of finite groups.

2 The principal series of $\Delta(27)$ and $\Delta(54)$

2.1 The structure of $\Delta(27)$ and its irreducible representations

The three exceptional groups $\Sigma(36 \times 3)$, $\Sigma(72 \times 3)$ and $\Sigma(216 \times 3)$, which we will discuss in this paper, all contain the generators C and E —see table 1, therefore,

$$\Delta(27) = \langle\langle C, E \rangle\rangle \quad (14)$$

is a subgroup of these three groups. We will later see that $\Delta(27)$ and also $\Delta(54)$ are even normal subgroups of $\Sigma(36 \times 3)$, $\Sigma(72 \times 3)$ and $\Sigma(216 \times 3)$.

With the relations

$$C^{-1}EC = \omega E, \quad E^{-1}CE = \omega^2 C, \quad (15)$$

the reordering relations

$$EC = \omega CE, \quad E^2C = \omega^2 CE^2, \quad EC^2 = \omega^2 C^2 E, \quad E^2C^2 = \omega C^2 E^2, \quad (16)$$

the definition

$$\mathcal{Z} \equiv \{\mathbb{1}, \omega \mathbb{1}, \omega^2 \mathbb{1}\} \quad (17)$$

and the abbreviation $\mathcal{Z} \cdot g \equiv \{g, \omega g, \omega^2 g\}$, it is easy to write down the conjugacy classes:

$$\begin{aligned} \mathcal{C}_1 &= \{\mathbb{1}\}, & \mathcal{C}_2 &= \{\omega \mathbb{1}\}, & \mathcal{C}_3 &= \{\omega^2 \mathbb{1}\}, \\ \mathcal{C}_4 &= \mathcal{Z} \cdot C, & \mathcal{C}_5 &= \mathcal{Z} \cdot C^2, & \mathcal{C}_6 &= \mathcal{Z} \cdot E, & \mathcal{C}_7 &= \mathcal{Z} \cdot E^2, \\ \mathcal{C}_8 &= \mathcal{Z} \cdot CE, & \mathcal{C}_9 &= \mathcal{Z} \cdot C^2 E^2, & \mathcal{C}_{10} &= \mathcal{Z} \cdot C^2 E, & \mathcal{C}_{11} &= \mathcal{Z} \cdot CE^2. \end{aligned} \quad (18)$$

Note that \mathcal{Z} consists of the elements of the center (6) of $SU(3)$. Furthermore, $\Delta(27)$ has the principal series

$$\{e\} \triangleleft \mathbb{Z}_3 \triangleleft \mathbb{Z}_3 \times \mathbb{Z}_3 \triangleleft \Delta(27) \quad (19)$$

where \mathbb{Z}_3 is given by equation (6), and

$$\mathbb{Z}_3 \times \mathbb{Z}_3 = \langle\langle \omega \mathbb{1} \rangle\rangle \times \langle\langle C \rangle\rangle. \quad (20)$$

The group $\mathbb{Z}_3 \times \mathbb{Z}_3$ consists of the set of elements

$$\mathcal{H} = \{\text{diag}(\omega^a, \omega^b, \omega^{-a-b}) \mid a, b = 0, 1, 2\}. \quad (21)$$

$\Delta(27)$ (# C_k) ord(C_k)	\mathcal{C}_1 (1) 1	\mathcal{C}_2 (1) 3	\mathcal{C}_3 (1) 3	\mathcal{C}_4 (3) 3	\mathcal{C}_5 (3) 3	\mathcal{C}_6 (3) 3	\mathcal{C}_7 (3) 3	\mathcal{C}_8 (3) 3	\mathcal{C}_9 (3) 3	\mathcal{C}_{10} (3) 3	\mathcal{C}_{11} (3) 3
$\mathbf{1}^{(0,0)}$	1	1	1	1	1	1	1	1	1	1	1
$\mathbf{1}^{(0,1)}$	1	1	1	1	1	ω	ω^2	ω	ω^2	ω	ω^2
$\mathbf{1}^{(0,2)}$	1	1	1	1	1	ω^2	ω	ω^2	ω	ω^2	ω
$\mathbf{1}^{(1,0)}$	1	1	1	ω	ω^2	1	1	ω	ω^2	ω^2	ω
$\mathbf{1}^{(1,1)}$	1	1	1	ω	ω^2	ω	ω^2	ω^2	ω	1	1
$\mathbf{1}^{(1,2)}$	1	1	1	ω	ω^2	ω^2	ω	1	1	ω	ω^2
$\mathbf{1}^{(2,0)}$	1	1	1	ω^2	ω	1	1	ω^2	ω	ω	ω^2
$\mathbf{1}^{(2,1)}$	1	1	1	ω^2	ω	ω	ω^2	1	1	ω^2	ω
$\mathbf{1}^{(2,2)}$	1	1	1	ω^2	ω	ω^2	ω	ω	ω^2	1	1
$\mathbf{3}$	3	3ω	$3\omega^2$	0	0	0	0	0	0	0	0
$\mathbf{3}^*$	3	$3\omega^2$	3ω	0	0	0	0	0	0	0	0

Table 2: Character table of $\Delta(27)$.

From equations (14) and (20) it follows that

$$\Delta(27)/(\mathbb{Z}_3 \times \mathbb{Z}_3) = \langle\langle \mathcal{H}E \rangle\rangle, \quad \Delta(27)/\mathbb{Z}_3 = \langle\langle \mathcal{Z}C, \mathcal{Z}E \rangle\rangle. \quad (22)$$

The cosets $\mathcal{Z}C$ and $\mathcal{Z}E$ commute, as can be read off from the first relation in equation (16). Therefore, the factor groups of the principal series (19) can be written as

$$\Delta(27)/(\mathbb{Z}_3 \times \mathbb{Z}_3) \cong \mathbb{Z}_3, \quad \Delta(27)/\mathbb{Z}_3 \cong \mathbb{Z}_3 \times \mathbb{Z}_3. \quad (23)$$

As for the irreps of $\Delta(27)$, equation (22) immediately gives the one-dimensional irreps

$$\mathbf{1}^{(p,q)} : \quad C \rightarrow \omega^p, \quad E \rightarrow \omega^q \quad (p, q = 0, 1, 2). \quad (24)$$

Note that, apart from the real irrep $\mathbf{1}^{(0,0)}$, all one-dimensional irreps are pairwise complex conjugate to each other, e.g., $(\mathbf{1}^{(1,2)})^* = \mathbf{1}^{(2,1)}$. (This is a general feature of all complex one-dimensional irreps of any group.) Since the number of conjugacy classes of $\Delta(27)$ is $n_c = 11$ —see equation (18)—and we have already found nine irreps, there are two remaining ones. With the relation

$$\sum_{j=1}^{n_c} d_j^2 = \text{ord}(G), \quad (25)$$

where the d_j are the dimensions of the irreps and $\text{ord}(G)$ is the order of the group, *i.e.* the number of its elements, the remaining irreps have both dimension three. Evidently they are given by

$$\begin{aligned} \mathbf{3} : \quad & C \rightarrow C, \quad E \rightarrow E, \\ \mathbf{3}^* : \quad & C \rightarrow C^*, \quad E \rightarrow E. \end{aligned} \quad (26)$$

Now that we have found all classes and all irreps of $\Delta(27)$, we can write down the character table—see table 2. In that table, and also in all following character tables, the second line indicates the number of elements in each class, the third line the order of the elements in each class. The order of an element g of a group is defined as the smallest positive power m such that $g^m = e$ where e is the unit element.

One can ask the question whether the two normal subgroups occurring in the principal series (19) are *all* normal subgroups of $\Delta(27)$. To find all normal subgroups N of a group G one can apply the following rules:

1. The order of any subgroup N is a divisor of the order of G .
2. Any normal subgroup N of G must contain complete classes of G .

Applying these rules to $\Delta(27)$, one quickly finds one additional normal subgroup:

$$N = \langle \langle \omega \mathbb{1}, E \rangle \rangle. \quad (27)$$

Thus, we can write down a second principal series

$$\{e\} \triangleleft \mathbb{Z}_3 \triangleleft N \triangleleft \Delta(27). \quad (28)$$

Obviously, N is isomorphic to $\mathbb{Z}_3 \times \mathbb{Z}_3$, which is formalized by $V^{-1}EV = C$ with V given by equation (3f). One can show by a simple computation that the principal series of equation (19) can be obtained via a similarity transformation with V from the principal series (28). This is a special case of the Jordan–Hölder theorem [28] which says that all principal series of a group G are isomorphic.

2.2 The group $\Delta(54)$

The conjugacy classes: This group is defined by

$$\Delta(54) = \langle \langle C, E, B \rangle \rangle = \langle \langle C, E, V^2 \rangle \rangle. \quad (29)$$

It is equivalent to use B or V^2 because

$$B = EV^2 \quad \text{with} \quad V^2 = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{pmatrix}. \quad (30)$$

The following theorems will be helpful for finding the conjugacy classes of $\Delta(54)$ and of the other groups discussed in this paper.

2.1 Theorem. Let H be a proper normal subgroup of G and $b \in G$ such that $b \notin H$. If C_k is a conjugacy class of H , then either $bC_kb^{-1} = C_k$ or the intersection between bC_kb^{-1} and C_k is empty.

Proof: In order to prove this statement we will show that, if there is an $h \in C_k$ such that $bhb^{-1} = h' \in C_k$, then this relation holds for *all* elements of C_k . The argument goes as follows. Since C_k is a class of H , any element $\tilde{h} \in C_k$ can be written as $\tilde{h} = h_1 h h_1^{-1}$ with $h_1 \in H$. Therefore,

$$b\tilde{h}b^{-1} = (bh_1b^{-1})(bhb^{-1})(bh_1b^{-1})^{-1} = (bh_1b^{-1})h'(bh_1b^{-1})^{-1} \in C_k,$$

because $bh_1b^{-1} \in H$. Q.E.D.

2.2 Theorem. Let H be a proper normal subgroup of G such that $G/H \cong \mathbb{Z}_r$ ($r \geq 2$) and let Hb be a generator of G/H . Then every conjugacy class of G can be written in the form Sb^ν where S is a subset of H and $\nu \in \{0, 1, \dots, r-1\}$. The conjugacy classes of G which are subsets of H can be obtained from the conjugacy classes of H in the following way:

- i. C_k is a conjugacy class of H such that $bC_kb^{-1} = C_k$. In this case C_k is also a conjugacy class of G .
- ii. C_k is a conjugacy class of H with empty intersection between bC_kb^{-1} and C_k . Then the corresponding conjugacy class of G is obtained by $C_k \cup bC_kb^{-1} \cup \dots \cup b^{r-1}C_kb^{-(r-1)}$.

Proof: With the above assumptions, the set of elements of G can be written as a union of cosets:

$$G = H \cup Hb \cup Hb^2 \cup \dots \cup Hb^{r-1}.$$

Note that $b^r \in H$ but $b^{r-1} \notin H$. Obviously, $b(Hb^\nu)b^{-1} = Hb^\nu$ holds because H is a normal subgroup of G . For the same reason we deduce

$$h(Hb^\nu)h^{-1} = H(b^\nu h^{-1}b^{-\nu})b^\nu = Hb^\nu.$$

Therefore, the cosets H, Hb, \dots, Hb^{r-1} are invariant under G and any class of G must be contained in one of the cosets. This proves the first part of the theorem. The second part is a consequence of theorem 2.1. Q.E.D.

Now we observe $\Delta(54)/\Delta(27) \cong \mathbb{Z}_2$, which allows us to use theorem 2.2 for constructing the conjugacy classes of $\Delta(54)$ from those of $\Delta(27)$. The matrix B corresponds to the element b occurring in theorem 2.2. We denote the classes of $\Delta(54)$ by \mathcal{C}'_k . With

$$B^{-1}CB = \omega^2 C^2, \quad B^{-1}EB = E^2 \quad (31)$$

we readily find

$$\begin{aligned} \mathcal{C}'_1 &= \mathcal{C}_1, & \mathcal{C}'_2 &= \mathcal{C}_2, & \mathcal{C}'_3 &= \mathcal{C}_3, \\ \mathcal{C}'_4 &= \mathcal{C}_4 \cup \mathcal{C}_5, & \mathcal{C}'_5 &= \mathcal{C}_6 \cup \mathcal{C}_7, & \mathcal{C}'_6 &= \mathcal{C}_8 \cup \mathcal{C}_9, & \mathcal{C}'_7 &= \mathcal{C}_{10} \cup \mathcal{C}_{11}, \end{aligned} \quad (32)$$

which exhausts the elements of $\Delta(27)$. Then we first search for the class generated by B . The relations

$$C^{-1}BC = \omega^2 CB, \quad E^{-1}BE = EB, \quad (33)$$

together with equations (15) and (16) lead to

$$\begin{aligned} \mathcal{C}'_8 &= \{B, EB, E^2B, CE^2B, C^2E^2B, \omega CEB, \omega C^2B, \omega^2 CB, \omega^2 C^2EB\} \\ &= \{V^2, EV^2, E^2V^2, CV^2, C^2V^2, \omega CE^2V^2, \omega C^2EV^2, \omega^2 CEV^2, \omega^2 C^2E^2V^2\}. \end{aligned} \quad (34)$$

It is then obvious that the missing conjugacy classes are given by

$$\mathcal{C}'_9 = \omega \mathcal{C}'_8, \quad \mathcal{C}'_{10} = \omega^2 \mathcal{C}'_8. \quad (35)$$

Thus there are ten classes of $\Delta(54)$ and, therefore, this group possesses ten inequivalent irreps.

Equations (15) and (31) establish the principal series (13) of $\Delta(54)$. Concerning additional normal subgroups not included in the principal series (13), the results developed at the end of section 2.1 apply for both $\Delta(27)$ and $\Delta(54)$, and the only additional proper normal subgroup is that of equation (27).

The factor groups: Now we want to discuss the factor groups of the principal series (13); this will be useful for constructing the irreps. First we have

$$\Delta(54)/\Delta(27) \cong \mathbb{Z}_2. \quad (36)$$

The generator of this \mathbb{Z}_2 is simply $\mathcal{D}B = \mathcal{D}V^2$, if we denote by \mathcal{D} the set of elements of $\Delta(27)$.

Using equation (21), we obtain

$$\Delta(54)/(\mathbb{Z}_3 \times \mathbb{Z}_3) = \langle \langle \mathcal{H}E, \mathcal{H}V^2 \rangle \rangle. \quad (37)$$

Remembering that the permutation group of three objects has the presentation

$$S_3: \quad a^2 = b^3 = (ab)^2 = e, \quad (38)$$

we find that the assignment

$$a \rightarrow \mathcal{H}V^2, \quad b \rightarrow \mathcal{H}E \quad (39)$$

constitutes an isomorphism because $(V^2)^2 = E^3 = (V^2E)^2 = \mathbb{1}$. The \mathbb{Z}_3 generated by b is a normal subgroup of S_3 because $aba^{-1} = aba = b^2$. Therefore, any element of S_3 can be written in the form $b^m a^n$ with $m = 0, 1, 2$ and $n = 0, 1$. In this way, S_3 has the multiplication law

$$(b^{m_1} a^{n_1}) (b^{m_2} a^{n_2}) = b^{m_1} (a^{n_1} b^{m_2} a^{-n_1}) a^{(n_1+n_2)} = b^{(m_1+2m_2n_1)} a^{(n_1+n_2)} \quad (40)$$

and S_3 can be considered as the semidirect product $\mathbb{Z}_3 \rtimes \mathbb{Z}_2$. The definition of a semidirect product of groups and an overview of its properties is given in appendix A. In summary we have proven that

$$\Delta(54)/(\mathbb{Z}_3 \times \mathbb{Z}_3) \cong S_3 \cong \mathbb{Z}_3 \rtimes \mathbb{Z}_2. \quad (41)$$

Next we consider

$$\Delta(54)/\mathbb{Z}_3 = \langle \langle \mathcal{Z}C, \mathcal{Z}E, \mathcal{Z}V^2 \rangle \rangle. \quad (42)$$

With $(V^2C)^2 = \mathbb{1}$, $CE = \omega^2 EC$ and the discussion related to S_3 , we notice that $\Delta(54)/\mathbb{Z}_3$ has a structure similar to S_3 , but with \mathbb{Z}_3 replaced by $\mathbb{Z}_3 \times \mathbb{Z}_3$. The assignment

$$a \rightarrow \mathcal{Z}V^2, \quad b \rightarrow \mathcal{Z}E, \quad c \rightarrow \mathcal{Z}C \quad (43)$$

allows to formulate the following presentation:

$$a^2 = b^3 = c^3 = (ab)^2 = (ac)^2 = e, \quad bc = cb. \quad (44)$$

Therefore, we conclude

$$\Delta(54)/\mathbb{Z}_3 \cong (\mathbb{Z}_3 \times \mathbb{Z}_3) \rtimes \mathbb{Z}_2. \quad (45)$$

This group is one of the three non-abelian groups with 18 elements, the other two being D_9 and $S_3 \times \mathbb{Z}_3$ [30].

The irreducible representations: Now we use the results of the factor groups to find all irreps of $\Delta(54)$. Equation (36) immediately leads to the two one-dimensional irreps

$$\begin{aligned} \mathbf{1} : & C \rightarrow 1, \quad E \rightarrow 1, \quad V^2 \rightarrow 1, \\ \mathbf{1}' : & C \rightarrow 1, \quad E \rightarrow 1, \quad V^2 \rightarrow -1. \end{aligned} \quad (46)$$

Next we search for irreps of $\Delta(54)$ by using equation (41). The one-dimensional irreps of S_3 are already taken care of by equation (46). It remains to consider its two-dimensional irrep which, by use of equation (39), is translated into an irrep of $\Delta(54)$. In a suitable basis, this irrep is given by the assignment

$$\mathbf{2} : \quad C \rightarrow \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad E \rightarrow \begin{pmatrix} \omega & 0 \\ 0 & \omega^2 \end{pmatrix}, \quad V^2 \rightarrow \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \quad (47)$$

Note that this irrep is real, with the representation matrix of V^2 acting as the transformation matrix which switches between the irrep and its complex conjugate. Moving to equation (45), the inequivalent irreps of $(\mathbb{Z}_3 \times \mathbb{Z}_3) \rtimes \mathbb{Z}_2$, which are not already contained in equations (46) and (47), are easily found with our knowledge of S_3 and usage of equation (43). There are three remaining inequivalent two-dimensional ones:

$$\begin{aligned} \mathbf{2}' : & C \rightarrow \begin{pmatrix} \omega & 0 \\ 0 & \omega^2 \end{pmatrix}, \quad E \rightarrow \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad V^2 \rightarrow \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \\ \mathbf{2}'' : & C \rightarrow \begin{pmatrix} \omega & 0 \\ 0 & \omega^2 \end{pmatrix}, \quad E \rightarrow \begin{pmatrix} \omega & 0 \\ 0 & \omega^2 \end{pmatrix}, \quad V^2 \rightarrow \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \\ \mathbf{2}''' : & C \rightarrow \begin{pmatrix} \omega & 0 \\ 0 & \omega^2 \end{pmatrix}, \quad E \rightarrow \begin{pmatrix} \omega^2 & 0 \\ 0 & \omega \end{pmatrix}, \quad V^2 \rightarrow \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \end{aligned} \quad (48)$$

Since $\Delta(54)$ has ten classes, four irreps are missing. They are easy to guess because the definition (29) itself constitutes a three-dimensional irrep, out of which we find three other ones by complex conjugation and by multiplication with $\mathbf{1}'$ of equation (46):

$$\begin{aligned} \mathbf{3} : & C \rightarrow C, \quad E \rightarrow E, \quad V^2 \rightarrow V^2, \\ (\mathbf{3})^* : & C \rightarrow C^*, \quad E \rightarrow E, \quad V^2 \rightarrow V^2, \\ \mathbf{3}' : & C \rightarrow C, \quad E \rightarrow E, \quad V^2 \rightarrow -V^2, \\ (\mathbf{3}')^* : & C \rightarrow C^*, \quad E \rightarrow E, \quad V^2 \rightarrow -V^2. \end{aligned} \quad (49)$$

The irreps we have constructed here agree with those given in [27]. The character table of $\Delta(54)$ is presented in table 3. This concludes the discussion of the irreps of $\Delta(27)$ and $\Delta(54)$ in the light of their principal series.

3 The group $\Sigma(36 \times 3)$

3.1 Definition and principal series

This group is generated by

$$\Sigma(36 \times 3) = \langle\langle C, E, V \rangle\rangle. \quad (50)$$

$\Delta(54)$ (# C_k) ord(C_k)	\mathcal{C}'_1	\mathcal{C}'_2	\mathcal{C}'_3	\mathcal{C}'_4	\mathcal{C}'_5	\mathcal{C}'_6	\mathcal{C}'_7	\mathcal{C}'_8	\mathcal{C}'_9	\mathcal{C}'_{10}
	(1)	(1)	(1)	(6)	(6)	(6)	(6)	(9)	(9)	(9)
	1	3	3	3	3	3	3	2	6	6
1	1	1	1	1	1	1	1	1	1	1
1'	1	1	1	1	1	1	1	-1	-1	-1
2	2	2	2	2	-1	-1	-1	0	0	0
2'	2	2	2	-1	2	-1	-1	0	0	0
2''	2	2	2	-1	-1	-1	2	0	0	0
2'''	2	2	2	-1	-1	2	-1	0	0	0
3	3	3ω	$3\omega^2$	0	0	0	0	-1	$-\omega$	$-\omega^2$
3'	3	3ω	$3\omega^2$	0	0	0	0	1	ω	ω^2
3*	3	$3\omega^2$	3ω	0	0	0	0	-1	$-\omega^2$	$-\omega$
(3')*	3	$3\omega^2$	3ω	0	0	0	0	1	ω^2	ω

Table 3: Character table of $\Delta(54)$.

It is easy to check that the generators of the group fulfill

$$C^3 = E^3 = V^4 = \mathbb{1}, \quad EV^{-1}CV = EVC^{-1}V^{-1} = (EC)^3 = \mathbb{1}. \quad (51)$$

According to GAP [29], these relations establish a presentation of $\Sigma(36 \times 3)$.

For the following considerations we need the set of relations

$$C^{-1}VC = C^2EV, \quad E^{-1}VE = \omega C^2E^2V \quad (52)$$

and

$$V^{-1}CV = E^2, \quad V^{-1}EV = C \quad (53)$$

With equations (30) and (53) and taking into account the definitions of \mathbb{Z}_3 , $\Delta(27)$ and $\Delta(54)$, one evidently finds the principal series

$$\{e\} \triangleleft \mathbb{Z}_3 \triangleleft \Delta(27) \triangleleft \Delta(54) \triangleleft \Sigma(36 \times 3). \quad (54)$$

Note that $\mathbb{Z}_3 \times \mathbb{Z}_3$ of equation (20) is not part of this principal series because it is not invariant under V —see equation (53). With the rules expounded at the end of section (2.1) one can show that the principal series (54) comprises all normal subgroups of $\Sigma(36 \times 3)$.

3.2 Conjugacy classes

The reordering relations (16) and

$$VC = EV, \quad VE = C^2V \quad (55)$$

allow to write every element of $\Sigma(36 \times 3)$ in the form

$$\omega^p C^q E^r V^s \quad \text{with} \quad p, q, r = 0, 1, 2, \quad s = 0, 1, 2, 3. \quad (56)$$

In order to find the classes of the group we will take advantage of this fact. Moreover, since $\Sigma(36 \times 3)/\Delta(54) \cong \mathbb{Z}_2$, we can make use of theorem 2.2. Then, with equations (15), (52) and (53), it is tedious but straightforward to find all the classes. We use the principal series (54) as a guideline for ordering the classes.

Each element of the center of the group is a class of its own, *i.e.*

$$C_1 = \{1\}, \quad C_2 = \{\omega 1\}, \quad C_3 = \{\omega^2 1\}. \quad (57)$$

Next we turn to the elements C and E which generate $\Delta(27)$. It turns out that these two elements generate the same class of $\Sigma(36 \times 3)$, whereas the product CE generates a different class:

$$C_4 = \{C, \omega C, \omega^2 C, C^2, \omega C^2, \omega^2 C^2, E, \omega E, \omega^2 E, E^2, \omega E^2, \omega^2 E^2\}, \quad (58)$$

$$C_5 = \{CE, \omega CE, \omega^2 CE, C^2 E, \omega C^2 E, \omega^2 C^2 E, CE^2, \omega CE^2, \omega^2 CE^2, C^2 E^2, \omega C^2 E^2, \omega^2 C^2 E^2\}. \quad (59)$$

These two classes have 12 elements each and together with $C_{1,2,3}$ they exhaust the subgroup $\Delta(27)$. Note that C_4 and C_5 are invariant under multiplication of their elements by ω , *i.e.* symbolically $\omega C_k = C_k$ for $k = 4, 5$. Now we go on to $\Delta(54)$ and consider the class generated by V^2 :

$$C_6 = \{V^2, EV^2, E^2 V^2, CV^2, C^2 V^2, \omega CE^2 V^2, \omega C^2 EV^2, \omega^2 CEV^2, \omega^2 C^2 E^2 V^2\}. \quad (60)$$

This class is identical with C'_8 of $\Delta(54)$. Obviously, in this case multiplication of the elements of C_6 by powers of ω gives two new classes:

$$C_7 = \omega C_6, \quad C_8 = \omega^2 C_6. \quad (61)$$

Eventually, we are left with the elements of $\Sigma(36 \times 3)$ which are not contained in $\Delta(54)$. The class generated by V is obtained as

$$C_9 = \{V, \omega CV, \omega C^2 V, \omega EV, \omega E^2 V, C^2 EV, CE^2 V, \omega CEV, \omega C^2 E^2 V\}, \quad (62)$$

which immediately results in the two further classes

$$C_{10} = \omega C_9, \quad C_{11} = \omega^2 C_9. \quad (63)$$

Finally, we are left with

$$C_{12} = \{V^3, \omega^2 CV^3, \omega^2 C^2 V^3, \omega^2 EV^3, \omega^2 E^2 V^3, \omega^2 C^2 EV^3, \omega^2 CE^2 V^3, CEV^3, C^2 E^2 V^3\} \quad (64)$$

and

$$C_{13} = \omega C_{12}, \quad C_{14} = \omega^2 C_{12}. \quad (65)$$

Noting that $V^3 = V^\dagger = V^*$, we see that C_{12} can be obtained from C_9 by complex conjugation.

3.3 Factor groups and irreps

Finding the factor groups G/G_k —see introduction—with respect to the principal series (54) is almost trivial:

$$\Sigma(36 \times 3)/\Delta(54) \cong \mathbb{Z}_2, \quad \Sigma(36 \times 3)/\Delta(27) \cong \mathbb{Z}_4, \quad \Sigma(36 \times 3)/\mathbb{Z}_3 \equiv \Sigma(36). \quad (66)$$

The first two relations follow from the property of the generator V . As discussed in the introduction, the last relation is simply the definition of $\Sigma(36)$.

From the first two factors in equation (66) it follows immediately that there are four one-dimensional irreps, characterized by

$$\mathbf{1}^{(p)} : \quad C \rightarrow 1, \quad E \rightarrow 1, \quad V \rightarrow i^p \quad (p = 0, 1, 2, 3). \quad (67)$$

Next we move to the group $\Sigma(36)$ whose elements consist of the cosets $g\mathcal{Z}$, where \mathcal{Z} defined in equation (17) contains the elements of the center of $\Sigma(36 \times 3)$ and $g \in \Sigma(36 \times 3)$. Using the results of subsection 3.2, it is straightforward to find the classes \tilde{C}_k of $\Sigma(36)$ from those of $\Sigma(36 \times 3)$:

$$\begin{aligned} \tilde{C}_1 &= \mathcal{Z}, \\ \tilde{C}_2 &= C_4\mathcal{Z}, \\ \tilde{C}_3 &= C_5\mathcal{Z}, \\ \tilde{C}_4 &= C_6\mathcal{Z} \equiv C_7\mathcal{Z} \equiv C_8\mathcal{Z}, \\ \tilde{C}_5 &= C_9\mathcal{Z} \equiv C_{10}\mathcal{Z} \equiv C_{11}\mathcal{Z}, \\ \tilde{C}_6 &= C_{12}\mathcal{Z} \equiv C_{13}\mathcal{Z} \equiv C_{14}\mathcal{Z}. \end{aligned} \quad (68)$$

Therefore, $\Sigma(36)$ possesses six irreps. These are also irreps of $\Sigma(36 \times 3)$ such that the center of $\Sigma(36 \times 3)$ is mapped onto unity. This is the case with the four one-dimensional irreps of equation (67), therefore, we already know four of the six irreps of $\Sigma(36)$. Denoting the dimensions of the remaining two irreps of $\Sigma(36)$ by d and d' , equation (25) tells us that

$$4 \times 1^2 + d^2 + d'^2 = 36. \quad (69)$$

This equation has a single solution for the two dimensions, namely

$$d = d' = 4. \quad (70)$$

We denote the corresponding irreps by $\mathbf{4}$ and $\mathbf{4}'$.

Since $E^{-1}C^{-1}EC = \omega \mathbb{1} \in \mathbb{Z}_3$ which is mapped onto unity for $\Sigma(36)$, we find that C and E commute in the four-dimensional irreps. Therefore, in these irreps we can adopt a basis where both C and E are diagonal. Equation (53) gives $V^{-2}CV^2 = C^2$ and $V^{-2}EV^2 = E^2$. Thus C and C^2 are both diagonal and have the same eigenvalues. Since $C^3 = \mathbb{1}$, the eigenvalues must be powers of ω . Up to reorderings there are two solutions for C , namely

$$C \rightarrow \text{diag}(1, \omega, 1, \omega^2) \quad \text{and} \quad \text{diag}(\omega, \omega, \omega^2, \omega^2). \quad (71)$$

$\Sigma(36)$ (# C_k) ord(C_k)	\tilde{C}_1 (1)	\tilde{C}_2 (4)	\tilde{C}_3 (4)	\tilde{C}_4 (9)	\tilde{C}_5 (9)	\tilde{C}_6 (9)
$\mathbf{1}^{(1)}$	1	1	1	1	1	1
$\mathbf{1}^{(2)}$	1	1	1	-1	i	$-i$
$\mathbf{1}^{(3)}$	1	1	1	1	-1	-1
$\mathbf{1}^{(4)}$	1	1	1	-1	$-i$	i
$\mathbf{4}$	4	1	-2	0	0	0
$\mathbf{4}'$	4	-2	1	0	0	0

Table 4: Character table of $\Sigma(36)$.

The same applies to the representation of E , only the ordering must be different in order to ensure that C and E are represented differently. A method for finding explicit realizations of the four-dimensional irreps is presented in appendix C. The result is given by

$$\mathbf{4} : C \rightarrow \text{diag}(1, \omega, 1, \omega^2), \quad E \rightarrow \text{diag}(\omega, 1, \omega^2, 1), \quad V \rightarrow \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix} \quad (72)$$

and

$$\mathbf{4}' : C \rightarrow \text{diag}(\omega, \omega, \omega^2, \omega^2), \quad E \rightarrow \text{diag}(\omega, \omega^2, \omega^2, \omega), \quad V \rightarrow \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix}. \quad (73)$$

One can check, for instance, that the relations (51) are fulfilled.

Up to now we have found six irreps of $\Sigma(36 \times 3)$. Since this group possesses 14 classes, there are eight remaining irreps. We guess that these have all dimensions three because then

$$4 \times 1^2 + 2 \times 4^2 + 8 \times 3^2 = 108. \quad (74)$$

This is indeed easily proven. Noting that the only 3×3 matrix which commutes with both C and E is proportional to the unit matrix, *i.e.* C and E generate a three-dimensional irrep of $\Delta(27)$, it is obvious that the defining representation (50) of $\Sigma(36 \times 3)$ is irreducible as well. With these considerations we use the one-dimensional irreps and complex conjugation to construct the eight three-dimensional irreps:

$$\begin{aligned} \mathbf{3}^{(p)} : & \quad C \rightarrow C, \quad E \rightarrow E, \quad V \rightarrow i^p V, \\ (\mathbf{3}^{(p)})^* : & \quad C \rightarrow C^*, \quad E \rightarrow E, \quad V \rightarrow (-i)^p V^*, \end{aligned} \quad (75)$$

where $p = 0, 1, 2, 3$. Therefore, $\mathbf{3}^{(p)} \equiv \mathbf{1}^{(p)} \otimes \mathbf{3}^{(0)}$. The character table of $\Sigma(36 \times 3)$ is given in table 5. The vertical lines in this table indicate the classes which merge into one class when one makes the transition from $\Sigma(36 \times 3)$ to $\Sigma(36)$.

[illegible]

4 The group $\Sigma(72 \times 3)$

4.1 Definition and principal series

The group is generated by

$$\Sigma(72 \times 3) = \langle\langle C, E, V, X \rangle\rangle. \quad (76)$$

Obviously, $\Sigma(36 \times 3)$ is a subgroup of $\Sigma(72 \times 3)$. Let us investigate the properties of the new generator X . First of all we find

$$X^2 = C^2V^2 \in \Sigma(36 \times 3), \quad X^4 = \mathbb{1}. \quad (77)$$

For the computation of the classes of $\Sigma(72 \times 3)$ we need

$$XCX^{-1} = C^2E, \quad XEX^{-1} = \omega^2CE, \quad XVX^{-1} = \omega^2EV^3 \quad (78)$$

and

$$C^{-1}XC = CEX, \quad E^{-1}XE = \omega CX, \quad V^{-1}XV = \omega^2CV^2X. \quad (79)$$

From equation (78) we read off that the principal series of $\Sigma(72 \times 3)$ is given by equation (11).

One can ask the question if the four normal subgroups occurring in the principal series (11) comprise all normal subgroups of $\Sigma(72 \times 3)$. The answer is no. Laborious application of the procedure outlined at the end of section 2.1 yields one more normal subgroup given by

$$N = \langle\langle C, E, X \rangle\rangle. \quad (80)$$

Performing a similarity transformation with the matrix D of equation (3c), we obtain

$$D^{-1}CD = C, \quad D^{-1}ED = CE, \quad D^{-1}XD = V, \quad (81)$$

which demonstrates that N is isomorphic to $\Sigma(36 \times 3)$. If we replace in the principal series (11) the subgroup $\Sigma(36 \times 3)$ by N , we obtain an isomorphic principal series with the isomorphism given by the transformation (81); this is again a manifestation of the Jordan–Hölder theorem.

4.2 Conjugacy classes

With the reordering relations

$$XC = C^2EX, \quad XE = \omega^2CEX, \quad XV = \omega^2EV^3X. \quad (82)$$

and with equation (77) we conclude that every element of $\Sigma(72 \times 3)$ can be written as

$$\omega^p C^q E^r V^s X^t \quad \text{with} \quad p, q, r = 0, 1, 2, \quad s = 0, 1, 2, 3, \quad t = 0, 1. \quad (83)$$

With theorem 2.2 and equation (78) we first search for the classes C'_k of $\Sigma(72 \times 3)$ which are subsets of $\Sigma(36 \times 3)$. Going through the classes C_j ($j = 1, \dots, 14$) of $\Sigma(36 \times 3)$, one

Classes	Number of elements
$C'_1 = C_1 = C_1$	1
$C'_2 = \omega C'_1$	1
$C'_3 = \omega^2 C'_1$	1
$C'_4 = C_4 \cup C_5 = C_C$	24
$C'_5 = C_6 = C_{V^2}$	9
$C'_6 = \omega C'_5$	9
$C'_7 = \omega^2 C'_5$	9
$C'_8 = C_9 \cup C_{12} = C_V$	18
$C'_9 = \omega C'_8$	18
$C'_{10} = \omega^2 C'_8$	18
$C'_{11} = C_X$	18
$C'_{12} = \omega C_X$	18
$C'_{13} = \omega^2 C_X$	18
$C'_{14} = C_{VX}$	18
$C'_{15} = \omega C'_{14}$	18
$C'_{16} = \omega^2 C'_{14}$	18
Total number of elements	216

Table 6: The conjugacy classes of $\Sigma(72 \times 3)$.

derives, with some patience, the upper part of table 6. The remaining classes must have elements with the standard form $\omega^p C^q E^r V^s X$. Let us start with the class which contains X :

$$C_X = \{X, \omega^2 V^2 X, \omega^2 E^2 V^2 X, \omega^2 C^2 E V^2 X, \omega C^2 X, \omega E V^2 X, C^2 V^2 X, \omega C X, \omega^2 C V^2 X, \omega^2 C E^2 V^2 X, \omega E X, \omega C E V^2 X, \omega C^2 E X, C^2 E^2 V^2 X, C E^2 X, \omega^2 C^2 E^2 X, C E X, E^2 X\}. \quad (84)$$

Since $\omega X, \omega^2 X \notin C_X$, we obtain the next two classes as

$$C_{\omega X} = \omega C_X, \quad C_{\omega^2 X} = \omega^2 C_X. \quad (85)$$

The simplest element that is not contained in the classes already listed is VX . Its associated class is given by

$$C_{VX} = \{VX, C V^3 X, \omega C E V^3 X, \omega^2 C^2 E V^3 X, \omega^2 E V X, \omega^2 C^2 V X, C E^2 V^3 X, \omega C^2 E^2 V^3 X, \omega^2 C E V X, \omega C E^2 V X, \omega E^2 V^3 X, \omega^2 C V X, C^2 E V X, E^2 V X, \omega C^2 V^3 X, \omega V^3 X, C^2 E^2 V X, \omega^2 E V^3 X\}. \quad (86)$$

Since $\omega VX, \omega^2 VX \notin C_{VX}$, the remaining two classes are

$$C_{\omega VX} = \omega C_{VX}, \quad C_{\omega^2 VX} = \omega^2 C_{VX}. \quad (87)$$

The conjugacy classes we have found are listed in table 6, from where we also read off that we have accommodated all elements of $\Sigma(72 \times 3)$. Therefore, the set of classes listed

in table 6 is complete. We have also indicated one member of each class. Now we face the task of finding all 16 irreps of $\Sigma(72 \times 3)$, which will be facilitated by finding first the factor groups.

4.3 The factor groups

The factor groups corresponding to the principal series (11) are given by

$$\Sigma(72 \times 3)/\Sigma(36 \times 3) \cong \mathbb{Z}_2, \quad \Sigma(72 \times 3)/\Delta(54) \cong \mathbb{Z}_2 \times \mathbb{Z}_2, \quad \Sigma(72 \times 3)/\Delta(27) \cong Q_8. \quad (88)$$

In addition, there is the factor group $\Sigma(72 \times 3)/\mathbb{Z}_3 \equiv \Sigma(72)$. The first statement of equation (88) follows from equation (77). In order to prove the second statement, we denote the set of matrices of $\Delta(54)$ by $\bar{\mathcal{D}}$. Then,

$$\Sigma(72 \times 3)/\Delta(54) = \langle \langle \bar{\mathcal{D}}V, \bar{\mathcal{D}}X \rangle \rangle. \quad (89)$$

Because of

$$VXV^{-1} = \omega EV^2X \quad \text{with} \quad \omega EV^2 \in \bar{\mathcal{D}}, \quad (90)$$

it follows that $\bar{\mathcal{D}}V$ and $\bar{\mathcal{D}}X$ commute. Moreover, since V^2 and X^2 are both elements of $\bar{\mathcal{D}}$, the right-hand side of equation (89) is isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2$, which is Klein's four-group.

The case of $\Sigma(72 \times 3)/\Delta(27)$ is more involved. We denote the set of matrices of $\Delta(27)$ by \mathcal{D} . Therefore, we find

$$\Sigma(72 \times 3)/\Delta(27) = \langle \langle \mathcal{D}V, \mathcal{D}X \rangle \rangle. \quad (91)$$

From equation (77) we deduce

$$(\mathcal{D}V)^2 = (\mathcal{D}X)^2 = \mathcal{D}V^2, \quad (\mathcal{D}V)^4 = (\mathcal{D}X)^4 = \mathcal{D}. \quad (92)$$

Moreover, equation (77) tells us that $\mathcal{D}V^2$ belongs to the center of $\Sigma(72 \times 3)/\Delta(27)$. Application of equation (90) leads to

$$(\mathcal{D}V)(\mathcal{D}X) = (\mathcal{D}V^2)(\mathcal{D}X)(\mathcal{D}V). \quad (93)$$

This suggests a mapping from $\mathcal{D}V, \mathcal{D}X, \mathcal{D}V^2$ to $i\sigma_1, i\sigma_2$ and $-\mathbb{1}_2$, respectively, with the Pauli matrices σ_k ($k = 1, 2, 3$). Indeed, with formulas (92) and (93) one can show that

$$\begin{aligned} \mathcal{D} &\rightarrow \mathbb{1}_2, & \mathcal{D}V &\rightarrow i\sigma_1, & \mathcal{D}X &\rightarrow i\sigma_2, & \mathcal{D}V^3X &\rightarrow i\sigma_3, \\ \mathcal{D}V^2 &\rightarrow -\mathbb{1}_2, & \mathcal{D}V^3 &\rightarrow -i\sigma_1, & \mathcal{D}V^2X &\rightarrow -i\sigma_2, & \mathcal{D}VX &\rightarrow -i\sigma_3 \end{aligned} \quad (94)$$

constitutes an isomorphism between $\Sigma(72 \times 3)/\Delta(27)$ and the group

$$Q_8 = \{\pm \mathbb{1}_2, \pm i\sigma_1, \pm i\sigma_2, \pm i\sigma_3\}. \quad (95)$$

For general discussions, model building and references on the groups Q_{2n} see [31]. The group Q_8 has five classes:

$$C_1^q = \{\mathbb{1}_2\}, \quad C_2^q = \{-\mathbb{1}_2\}, \quad C_3^q = \{\pm i\sigma_1\}, \quad C_4^q = \{\pm i\sigma_2\}, \quad C_5^q = \{\pm i\sigma_3\}. \quad (96)$$

There are four one-dimensional irreps

$$\pm \mathbb{1}_2 \rightarrow 1, \quad \pm i\sigma_1 \rightarrow (-1)^p, \quad \pm i\sigma_2 \rightarrow (-1)^q, \quad \pm i\sigma_3 \rightarrow (-1)^{p+q} \quad (97)$$

with $p, q = 0, 1$. The remaining irrep of Q_8 is the defining two-dimensional irrep (95).

Q_8	C_1^q	C_2^q	C_3^q	C_4^q	C_5^q
$(\# C_k)$	(1)	(1)	(2)	(2)	(2)
$\text{ord}(C_k)$	1	2	4	4	4
$\mathbf{1}^{(0,0)}$	1	1	1	1	1
$\mathbf{1}^{(1,0)}$	1	1	-1	1	-1
$\mathbf{1}^{(0,1)}$	1	1	1	-1	-1
$\mathbf{1}^{(1,1)}$	1	1	-1	-1	1
2	2	-2	0	0	0

Table 7: Character table of Q_8 .

4.4 The irreps of $\Sigma(72 \times 3)$

The first two factor groups in equation (88) furnish the one-dimensional irreps of $\Sigma(72 \times 3)$:

$$\mathbf{1}^{(p,q)} : C \rightarrow 1, \quad E \rightarrow 1, \quad V \rightarrow (-1)^p, \quad X \rightarrow (-1)^q \quad (p, q = 0, 1). \quad (98)$$

These correspond to the one-dimensional irreps of the factor group Q_8 . The two-dimensional irrep of Q_8 can be translated according to equation (94) into an irrep of $\Sigma(72 \times 3)$ by

$$\mathbf{2} : C \rightarrow \mathbb{1}_2, \quad E \rightarrow \mathbb{1}_2, \quad V \rightarrow i\sigma_1, \quad X \rightarrow i\sigma_2. \quad (99)$$

The last factor group is $\Sigma(72)$. Its classes are almost trivially found from those of $\Sigma(72 \times 3)$:

$$\begin{aligned} \tilde{C}'_1 &= \mathcal{Z}, \\ \tilde{C}'_2 &= C'_4 \mathcal{Z}, \\ \tilde{C}'_3 &= C'_5 \mathcal{Z} \equiv C'_6 \mathcal{Z} \equiv C'_7 \mathcal{Z}, \\ \tilde{C}'_4 &= C'_8 \mathcal{Z} \equiv C'_9 \mathcal{Z} \equiv C'_{10} \mathcal{Z}, \\ \tilde{C}'_5 &= C'_{11} \mathcal{Z} \equiv C'_{12} \mathcal{Z} \equiv C'_{13} \mathcal{Z}, \\ \tilde{C}'_6 &= C'_{14} \mathcal{Z} \equiv C'_{15} \mathcal{Z} \equiv C'_{16} \mathcal{Z}. \end{aligned} \quad (100)$$

Since the one and two-dimensional irreps of $\Sigma(72 \times 3)$ which we have already constructed have a trivial center, there remains one irrep of $\Sigma(72)$. Because of $4 \times 1^2 + 2^2 + 8^2 = 72$, this irrep must have dimension eight. If we denote—for reasons to become clear soon—the defining irrep of $\Sigma(72 \times 3)$ by $\mathbf{3}^{(0,0)}$, this irrep **8** must be obtained by

$$\mathbf{3}^{(0,0)} \otimes (\mathbf{3}^{(0,0)})^* = \mathbf{1} \oplus \mathbf{8}. \quad (101)$$

For $\Sigma(36 \times 3)$ the eight-dimensional representation decayed into two four-dimensional irreps—see appendix C. However, $\Sigma(72 \times 3)$ has one generator more and the **8** is irreducible, as one can easily show. The character of the **8** is thus given by

$$\chi_{\mathbf{8}} = |\chi_{\mathbf{3}^{(0,0)}}|^2 - 1, \quad (102)$$

which allows to complete the character table of $\Sigma(72)$.

$\Sigma(72)$ (# C_k) $\text{ord}(C_k)$	\tilde{C}'_1 (1)	\tilde{C}'_2 (8)	\tilde{C}'_3 (9)	\tilde{C}'_4 (18)	\tilde{C}'_5 (18)	\tilde{C}'_6 (18)
	1	3	2	4	4	4
$\mathbf{1}^{(1)}$	1	1	1	1	1	1
$\mathbf{1}^{(2)}$	1	1	1	-1	1	-1
$\mathbf{1}^{(3)}$	1	1	1	1	-1	-1
$\mathbf{1}^{(4)}$	1	1	1	-1	-1	1
$\mathbf{2}$	2	2	-2	0	0	0
$\mathbf{8}$	8	-1	0	0	0	0

Table 8: Character table of $\Sigma(72)$.

Now we have exhausted the factor groups. The ten remaining irreps must be faithful. We readily find eight three-dimensional irreps by multiplying the defining irrep with the one-dimensional irreps and by complex conjugation:

$$\mathbf{3}^{(p,q)} \equiv \mathbf{1}^{(p,q)} \otimes \mathbf{3}^{(0,0)}, \quad (\mathbf{3}^{(p,q)})^* \equiv \mathbf{1}^{(p,q)} \otimes (\mathbf{3}^{(0,0)})^*. \quad (103)$$

Explicitly, the generators of $\Sigma(72 \times 3)$ are represented by

$$\begin{aligned} \mathbf{3}^{(p,q)} : \quad & C \rightarrow C, \quad E \rightarrow E, \quad V \rightarrow (-1)^p V, \quad X \rightarrow (-1)^q X, \\ (\mathbf{3}^{(p,q)})^* : \quad & C \rightarrow C^*, \quad E \rightarrow E, \quad V \rightarrow (-1)^p V^*, \quad X \rightarrow (-1)^q X^*. \end{aligned} \quad (104)$$

There are two missing irreps with dimensions d and d' , which fulfill $d^2 + d'^2 = 72$. The solution of this equation is unique: $d = d' = 6$. Therefore, the six-dimensional irreps are obtained by the tensor product of the defining irrep with itself. Again, just as for the $\mathbf{8}$, the $\mathbf{6}$ and $\mathbf{6}^*$ are irreducible because of the generator X in addition to those of $\Sigma(36 \times 3)$. We define the $\mathbf{6}^*$ by

$$\mathbf{3}^{(0,0)} \otimes \mathbf{3}^{(0,0)} = (\mathbf{3}^{(0,0)})^* \oplus \mathbf{6}^* \quad (105)$$

and the $\mathbf{6}$ by its complex conjugate. This completes the construction of all irreps of $\Sigma(72 \times 3)$. With equation (105), the character of the $\mathbf{6}^*$ is then given by

$$\chi_{\mathbf{6}^*} = (\chi_{\mathbf{3}^{(0,0)}})^2 - (\chi_{\mathbf{3}^{(0,0)}})^*. \quad (106)$$

With $\chi_{\mathbf{6}} = (\chi_{\mathbf{6}^*})^*$, we can complete the character table of $\Sigma(72 \times 3)$ —see table 9.

5 The group $\Sigma(216 \times 3)$

5.1 Definition, principal series and conjugacy classes

The so-called Hessian group $\Sigma(216 \times 3)$ is generated by

$$\Sigma(216 \times 3) = \langle\langle C, E, V, D \rangle\rangle. \quad (107)$$

Table 9: Character table of $\Sigma(72 \times 3)$.

$\Sigma(72 \times 3)$ (# C_k) $\text{ord}(C_k)$	C'_1 (1) 1	C'_2 (1) 3	C'_3 (1) 3	C'_4 (24) 3	C'_5 (9) 2	C'_6 (9) 6	C'_7 (9) 6	C'_8 (18) 4	C'_9 (18) 12	C'_{10} (18) 12	C'_{11} (18) 4	C'_{12} (18) 12	C'_{13} (18) 12	C'_{14} (18) 12	C'_{15} (18) 12	C'_{16} (18) 4
$\mathbf{1}^{(0,0)}$	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
$\mathbf{1}^{(1,0)}$	1	1	1	1	1	1	1	-1	-1	-1	1	1	1	-1	-1	-1
$\mathbf{1}^{(0,1)}$	1	1	1	1	1	1	1	1	1	1	-1	-1	-1	-1	-1	-1
$\mathbf{1}^{(1,1)}$	1	1	1	1	1	1	1	-1	-1	-1	-1	-1	-1	1	1	1
$\mathbf{2}$	2	2	2	2	-2	-2	-2	0	0	0	0	0	0	0	0	0
$\mathbf{3}^{(0,0)}$	3	3ω	$3\omega^2$	0	-1	$-\omega$	$-\omega^2$	1	ω	ω^2	1	ω	ω^2	ω	ω^2	1
$\mathbf{3}^{(1,0)}$	3	3ω	$3\omega^2$	0	-1	$-\omega$	$-\omega^2$	-1	$-\omega$	$-\omega^2$	1	ω	ω^2	$-\omega$	$-\omega^2$	-1
$\mathbf{3}^{(0,1)}$	3	3ω	$3\omega^2$	0	-1	$-\omega$	$-\omega^2$	1	ω	ω^2	-1	$-\omega$	$-\omega^2$	$-\omega$	$-\omega^2$	-1
$\mathbf{3}^{(1,1)}$	3	3ω	$3\omega^2$	0	-1	$-\omega$	$-\omega^2$	-1	$-\omega$	$-\omega^2$	-1	$-\omega$	$-\omega^2$	ω	ω^2	1
$(\mathbf{3}^{(0,0)})^*$	3	$3\omega^2$	3ω	0	-1	$-\omega^2$	$-\omega$	1	ω^2	ω	1	ω^2	ω	ω^2	ω	1
$(\mathbf{3}^{(1,0)})^*$	3	$3\omega^2$	3ω	0	-1	$-\omega^2$	$-\omega$	-1	$-\omega^2$	$-\omega$	1	ω^2	ω	$-\omega^2$	$-\omega$	-1
$(\mathbf{3}^{(0,1)})^*$	3	$3\omega^2$	3ω	0	-1	$-\omega^2$	$-\omega$	1	ω^2	ω	-1	$-\omega^2$	$-\omega$	$-\omega^2$	$-\omega$	-1
$(\mathbf{3}^{(1,1)})^*$	3	$3\omega^2$	3ω	0	-1	$-\omega^2$	$-\omega$	-1	$-\omega^2$	$-\omega$	-1	$-\omega^2$	$-\omega$	ω^2	ω	1
$\mathbf{6}$	6	6ω	$6\omega^2$	0	2	2ω	$2\omega^2$	0	0	0	0	0	0	0	0	0
$\mathbf{6}^*$	6	$6\omega^2$	6ω	0	2	$2\omega^2$	2ω	0	0	0	0	0	0	0	0	0
$\mathbf{8}$	8	8	8	-1	0	0	0	0	0	0	0	0	0	0	0	0

Because of $X = DVD^{-1}$, also X is an element of $\Sigma(216 \times 3)$. In order to prove the principal series (12), we use the following relations:

$$D^{-1}CD = C, \quad D^{-1}ED = CE \Rightarrow \Delta(27) \triangleleft \Sigma(216 \times 3), \quad (108a)$$

$$D^{-1}V^2D = CV^2 \Rightarrow \Delta(54) \triangleleft \Sigma(216 \times 3), \quad (108b)$$

$$D^{-1}VD = \omega^2 CV^3 X, \quad D^{-1}XD = V \Rightarrow \Sigma(72 \times 3) \triangleleft \Sigma(216 \times 3). \quad (108c)$$

With our knowledge about the principal series of $\Sigma(72 \times 3)$ this proves equation (12), the principal series of $\Sigma(216 \times 3)$. The group $\Sigma(36 \times 3)$ is not a member of this principal series because of the first relation in equation (108c); moreover, this is also clear from the discussion at the end of section 4.1.

Further relations for the determination of the classes of $\Sigma(216 \times 3)$ are

$$C^{-1}DC = D, \quad E^{-1}DE = \omega C^2 D, \quad V^{-1}DV = V^3 DV = V^3 XD. \quad (109)$$

Note that without X the expression $V^3 DV$ cannot be reordered anymore. The same applies to $D^{-1}VD = CV^3 DVD^2 = \omega^2 CV^3 X$. This is the reason that the elements of $\Sigma(216 \times 3)$ cannot be written as $\omega^\alpha C^\beta E^\gamma V^\delta D^\epsilon$ with $\alpha, \beta, \gamma = 0, 1, 2$, $\delta = 0, 1, 2, 3$ and $\epsilon = 0, 1, 2$. (Note that $D^3 = \omega^2 \mathbb{1}$.) Actually, if this were the case, the number of elements of this group would be too small: $3 \times 3 \times 3 \times 4 \times 3 = 324$. However, every element $g \in \Sigma(216 \times 3)$ can be written as

$$g = \omega^p C^q E^r V^s X^t D^u \quad \text{with} \quad p, q, r, u = 0, 1, 2, \quad s = 0, 1, 2, 3, \quad t = 0, 1, \quad (110)$$

which gives the correct number of elements.

In order to find the conjugacy classes of $\Sigma(216 \times 3)$ one could use theorems 2.1 and 2.2 and the conjugacy classes of $\Sigma(72 \times 3)$. However, since $\Sigma(216 \times 3)$ has 648 elements, we use GAP [29] to find the classes and confine ourselves to checking the result. The list of classes is displayed in table 10. For the classes C_k'' of $\Sigma(216 \times 3)$ which are subsets of $\Sigma(72 \times 3)$ we have indicated the classes C_l' of $\Sigma(72 \times 3)$ they consist of. Furthermore, we have characterized every C_k'' by one element $g \in C_k''$.

One can show, by proceeding as described at the end of section 2.1, that $\Sigma(216 \times 3)$ has no other proper normal subgroups than those appearing in its principal series.

5.2 The factor groups

In order to characterize the factor groups in the principal series (12), we first have to discuss the groups T' and A_4 , as will shortly become clear.

According to GAP, the group T' , the double covering group of A_4 , has a presentation with two generators a, b as

$$T' : \quad a^4 = a^2 b^{-3} = (ab)^3 = e. \quad (111)$$

Obviously, the element

$$v \equiv a^2 = b^3 \quad (112)$$

Class	Number of elements
$C_1'' = C_1' = C_1$	1
$C_2'' = \omega C_1''$	1
$C_3'' = \omega^2 C_1''$	1
$C_4'' = C_4' = C_C$	24
$C_5'' = C_5' = C_{V^2}$	9
$C_6'' = \omega C_5''$	9
$C_7'' = \omega^2 C_5''$	9
$C_8'' = C_8' \cup C_{11}' \cup C_{16}' = C_V$	54
$C_9'' = \omega C_8''$	54
$C_{10}'' = \omega^2 C_8''$	54
$C_{11}'' = C_{ED}$	72
$C_{12}'' = C_{ED^2}$	72
$C_{13}'' = C_D$	12
$C_{14}'' = \omega C_D$	12
$C_{15}'' = \omega^2 C_D$	12
$C_{16}'' = C_{D^2}$	12
$C_{17}'' = \omega C_{D^2}$	12
$C_{18}'' = \omega^2 C_{D^2}$	12
$C_{19}'' = C_{V^2 D}$	36
$C_{20}'' = \omega C_{V^2 D}$	36
$C_{21}'' = \omega^2 C_{V^2 D}$	36
$C_{22}'' = C_{VD^2}$	36
$C_{23}'' = \omega C_{VD^2}$	36
$C_{24}'' = \omega^2 C_{VD^2}$	36
Total number of elements	648

Table 10: The conjugacy classes of $\Sigma(216 \times 3)$.

belongs to the center of T' . From the presentation (111), the group A_4 is obtained by the restriction $v = a^2 = b^3 = e$, *i.e.*

$$A_4 : \quad a^2 = b^3 = (ab)^3 = e. \quad (113)$$

It is easy to show that Klein's four-group $\mathbb{Z}_2 \times \mathbb{Z}_2$ is a normal subgroup of A_4 generated by the commuting elements a and bab^{-1} . Therefore, A_4 has the structure $(\mathbb{Z}_2 \times \mathbb{Z}_2) \rtimes \mathbb{Z}_3$ where b generates the \mathbb{Z}_3 .

Alternatively, one can use the generators $s = a$, $t = ab$ for a presentation of T' . These fulfill [32]

$$T' : \quad s^4 = t^3 = (st)^3 = e. \quad (114)$$

In equation (114) the step from T' to A_4 requires $s^2 = e$.

Now we establish the isomorphisms

$$\Sigma(216 \times 3)/\Delta(27) \cong T', \quad \Sigma(216 \times 3)/\Delta(54) \cong A_4, \quad \Sigma(216 \times 3)/\Sigma(72 \times 3) \cong \mathbb{Z}_3. \quad (115)$$

With \mathcal{D} denoting again the set of elements of $\Delta(27)$, the assignment

$$s \rightarrow \mathcal{D}V, \quad t \rightarrow \mathcal{D}D \quad (116)$$

proves the isomorphism $\Sigma(216 \times 3)/\Delta(27) \cong T'$, because $V^4 = (VD)^3 = \mathbb{1}$ and $D^3 = \omega^2 \mathbb{1} \in \mathcal{D}$. Consequently, the second relation of equation (115) is proven by the assignment

$$s \rightarrow \bar{\mathcal{D}}V, \quad t \rightarrow \bar{\mathcal{D}}D, \quad (117)$$

where $\bar{\mathcal{D}}$ is the set of elements of $\Delta(54)$. Since $V^2 \in \bar{\mathcal{D}}$ and, therefore, $(\bar{\mathcal{D}}V)^2 = \bar{\mathcal{D}}$, the assignment (117) establishes the isomorphism between A_4 and $\Sigma(216 \times 3)/\Delta(54)$. The third relation in equation (115) follows trivially from

$$\Sigma(216 \times 3)/\Sigma(72 \times 3) = \langle \langle \tilde{\mathcal{D}}D \rangle \rangle, \quad (118)$$

where $\tilde{\mathcal{D}}$ denotes the sets of elements of $\Sigma(72 \times 3)$.

Now, for the purpose of applying the result to $\Sigma(216 \times 3)$, we develop the representation theory of A_4 and T' . The relationship between these group is the same as between $SO(3)$ and $SU(2)$. Given a rotation matrix $R \in SO(3)$ then there are exactly two $SU(2)$ matrices U which differ only in the overall sign such that

$$U (\vec{\sigma} \cdot \vec{x}) U^\dagger = \vec{\sigma} \cdot (R\vec{x}) \quad (119)$$

for all vectors $\vec{x} \in \mathbb{R}^3$. We use the notation $\vec{\sigma} \cdot \vec{x} \equiv \sum_{k=1}^3 \sigma_k x_k$ with the Pauli matrices σ_k . Vice versa, any $SU(2)$ matrix U induces a rotation on \mathbb{R}^3 via equation (119).

According to the definition of A_4 in section 1, this group is generated by the $SO(3)$ matrices

$$R_a = A, \quad R_b = E, \quad (120)$$

which fulfill all relations of the presentation of A_4 mentioned above. The corresponding matrices U_a and U_b have to be found through equation (119). In contrast to $SU(2)$, the signs of these matrices are fixed by the additional relations of the presentation of T' .

Firstly, U_a^2 is in the center, therefore, $U_a^2 = -\mathbb{1}_2 = U_b^3$ and, secondly, $(U_a U_b)^3 = \mathbb{1}_2$. The result of the computation is

$$U_a = i \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad U_b = \frac{1}{\sqrt{2}} \begin{pmatrix} \phi & \phi \\ -\phi^* & \phi^* \end{pmatrix} \quad \text{with} \quad \phi = e^{i\pi/4}. \quad (121)$$

Equations (116) and (117) suggest to use

$$U_t \equiv U_a U_b = \frac{1}{\sqrt{2}} \begin{pmatrix} -\phi & \phi \\ -\phi^* & -\phi^* \end{pmatrix} \quad (122)$$

instead of U_b . In order to construct all irreps of T' , it is useful to first compute its classes. With the relation of the presentation we obtain

$$\begin{aligned} C_1^t &= \{e\}, \\ C_2^t &= \{v\}, \\ s &\in C_3^t = \{a, va, bab^2, vbab^2, b^2ab, vb^2ab\}, \\ t &\in C_4^t = \{ab, ba, vb, aba\}, \\ t^2 &\in C_5^t = \{ab^2, b^2a, b^2, bab\}, \\ vt &\in C_6^t = \{vab, vba, b, vaba\}, \\ vt^2 &\in C_7^t = \{vab^2, vb^2a, vb^2, vbab\} \end{aligned} \quad (123)$$

Thus we know that T' has seven irreps. The one-dimensional irreps are given by

$$\mathbf{1}^{(p)} : \quad s \rightarrow 1, \quad t \rightarrow \omega^p \quad (p = 0, 1, 2). \quad (124)$$

We have already found a two-dimensional irrep given by U_a and U_b (or U_t) which we denote by $\mathbf{2}^{(0)}$. The other two-dimensional irreps are given by $\mathbf{2}^{(p)} = \mathbf{1}^{(p)} \otimes \mathbf{2}^{(0)}$. Therefore, we have

$$\mathbf{2}^{(p)} : \quad s \rightarrow U_a, \quad t \rightarrow \omega^p U_t \quad (p = 0, 1, 2). \quad (125)$$

There remains a three-dimensional irrep which, according to $T'/\mathbb{Z}_2 \cong A_4$, must be the one given by the matrices of A_4 :

$$\mathbf{3}^{(a)} : s \rightarrow R_a = \text{diag}(1, -1, -1), \quad t \rightarrow R_a R_b = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & -1 \\ -1 & 0 & 0 \end{pmatrix}. \quad (126)$$

Knowledge of all classes and irreps allows to write down the character table of T' —see table 11. Further information on this group can be found, for instance, in [32, 33].

Having complete information on T' , the discussion of A_4 is trivial. The classes of A_4 are obtained from those of T' by setting $v = e$ and deleting the elements and classes which occur twice:

$$\begin{aligned} C_1^a &= \{e\}, \\ s &\in C_2^a = \{a, bab^2, b^2ab\}, \\ t &\in C_3^a = \{ab, ba, b, aba\}, \\ t^2 &\in C_4^a = \{ab^2, b^2a, b^2, bab\} \end{aligned} \quad (127)$$

The irreps are given by equations (124) and (126). For the sake of completeness we present the character table of A_4 in table 12. An interesting feature is that $\mathbf{1}^{(p)} \otimes \mathbf{3}^{(a)} \cong \mathbf{3}^{(a)}$. This follows from the character table 12 due to $\chi_{\mathbf{1}^{(p)}} \cdot \chi_{\mathbf{3}^{(a)}} = \chi_{\mathbf{3}^{(a)}}$.

T'	C_1^t	C_2^t	C_3^t	C_4^t	C_5^t	C_6^t	C_7^t
(# C_k)	(1)	(1)	(6)	(4)	(4)	(4)	(4)
ord(C_k)	1	2	4	3	3	6	6
$\mathbf{1}^{(0)}$	1	1	1	1	1	1	1
$\mathbf{1}^{(1)}$	1	1	1	ω	ω^2	ω	ω^2
$\mathbf{1}^{(2)}$	1	1	1	ω^2	ω	ω^2	ω
$\mathbf{2}^{(0)}$	2	-2	0	-1	-1	1	1
$\mathbf{2}^{(1)}$	2	-2	0	$-\omega$	$-\omega^2$	ω	ω^2
$\mathbf{2}^{(2)}$	2	-2	0	$-\omega^2$	$-\omega$	ω^2	ω
$\mathbf{3}^{(a)}$	3	3	-1	0	0	0	0

Table 11: Character table of T' .

A_4	C_1^a	C_2^a	C_3^a	C_4^a
(# C_k)	(1)	(3)	(4)	(4)
ord(C_k)	1	2	3	3
$\mathbf{1}^{(0)}$	1	1	1	1
$\mathbf{1}^{(1)}$	1	1	ω	ω^2
$\mathbf{1}^{(2)}$	1	1	ω^2	ω
$\mathbf{3}^{(a)}$	3	-1	0	0

Table 12: Character table of A_4 .

$\Sigma(216)$ (# C_k) $\text{ord}(C_k)$	\tilde{C}_1'' (1)	\tilde{C}_2'' (8)	\tilde{C}_3'' (9)	\tilde{C}_4'' (54)	\tilde{C}_5'' (24)	\tilde{C}_6'' (24)	\tilde{C}_7'' (12)	\tilde{C}_8'' (12)	\tilde{C}_9'' (36)	\tilde{C}_{10}'' (36)
$\mathbf{1}^{(0)}$	1	1	1	1	1	1	1	1	1	1
$\mathbf{1}^{(1)}$	1	1	1	1	ω	ω^2	ω	ω^2	ω	ω^2
$\mathbf{1}^{(2)}$	1	1	1	1	ω^2	ω	ω^2	ω	ω^2	ω
$\mathbf{2}^{(0)}$	2	2	-2	0	-1	-1	-1	-1	1	1
$\mathbf{2}^{(1)}$	2	2	-2	0	$-\omega$	$-\omega^2$	$-\omega$	$-\omega^2$	ω	ω^2
$\mathbf{2}^{(2)}$	2	2	-2	0	$-\omega^2$	$-\omega$	$-\omega^2$	$-\omega$	ω^2	ω
$\mathbf{3}^{(a)}$	3	3	3	-1	0	0	0	0	0	0
$\mathbf{8}^{(0)}$	8	-1	0	0	-1	-1	2	2	0	0
$\mathbf{8}^{(1)}$	8	-1	0	0	$-\omega$	$-\omega^2$	2ω	$2\omega^2$	0	0
$\mathbf{8}^{(2)}$	8	-1	0	0	$-\omega^2$	$-\omega$	$2\omega^2$	2ω	0	0

Table 13: Character table of $\Sigma(216)$.

5.3 Irreps of $\Sigma(216 \times 3)$

The excursion into T' and A_4 nets us seven irreps of $\Sigma(216 \times 3)$, by utilizing equations (116), (117) and (118). The one-dimensional irreps are given by

$$\mathbf{1}^{(p)} : \quad C \rightarrow 1 \quad E \rightarrow 1, \quad V \rightarrow 1, \quad D \rightarrow \omega^p \quad (p = 0, 1, 2). \quad (128)$$

The three-dimensional irrep of A_4 (or T') [14] leads to

$$\mathbf{3}^{(a)} : \quad C \rightarrow \mathbb{1} \quad E \rightarrow \mathbb{1}, \quad V \rightarrow R_a, \quad D \rightarrow R_a R_b. \quad (129)$$

Finally, with the two-dimensional irreps of T' we find

$$\mathbf{2}^{(p)} : \quad C \rightarrow \mathbb{1}_2 \quad E \rightarrow \mathbb{1}_2, \quad V \rightarrow U_a, \quad D \rightarrow \omega^p U_t \quad (p = 0, 1, 2). \quad (130)$$

Until now we have exploited the first two factor groups of equation (115). It remains to discuss $\Sigma(216) = \Sigma(216 \times 3)/\mathbb{Z}_3$. Examining table 10 we find that the 24 classes of $\Sigma(216 \times 3)$ collapse into ten classes of $\Sigma(216)$:

$$\begin{aligned}
\tilde{C}_1'' &= \mathcal{Z}, \\
\tilde{C}_2'' &= C_4'' \mathcal{Z}, \\
\tilde{C}_3'' &= C_5'' \mathcal{Z} \equiv C_6'' \mathcal{Z} \equiv C_7'' \mathcal{Z}, \\
\tilde{C}_4'' &= C_8'' \mathcal{Z} \equiv C_9'' \mathcal{Z} \equiv C_{10}'' \mathcal{Z}, \\
\tilde{C}_5'' &= C_{11}'' \mathcal{Z}, \\
\tilde{C}_6'' &= C_{12}'' \mathcal{Z}, \\
\tilde{C}_7'' &= C_{13}'' \mathcal{Z} \equiv C_{14}'' \mathcal{Z} \equiv C_{15}'' \mathcal{Z}, \\
\tilde{C}_8'' &= C_{16}'' \mathcal{Z} \equiv C_{17}'' \mathcal{Z} \equiv C_{18}'' \mathcal{Z}, \\
\tilde{C}_9'' &= C_{19}'' \mathcal{Z} \equiv C_{20}'' \mathcal{Z} \equiv C_{21}'' \mathcal{Z}, \\
\tilde{C}_{10}'' &= C_{22}'' \mathcal{Z} \equiv C_{23}'' \mathcal{Z} \equiv C_{24}'' \mathcal{Z}.
\end{aligned} \quad (131)$$

Since the irreps in equations (128), (129) and (130) map the center \mathbb{Z}_3 of $\Sigma(216 \times 3)$ onto the unit matrix, these irreps provide seven irreps of $\Sigma(216)$. There are three remaining ones. We denote the defining irrep of $\Sigma(216 \times 3)$ by $\mathbf{3}^{(0)}$. Since in the tensor product

$$\mathbf{3}^{(0)} \otimes (\mathbf{3}^{(0)})^* = \mathbf{1}^{(0)} \oplus \mathbf{8}^{(0)} \quad (132)$$

the center is trivially represented, the $\mathbf{8}^{(0)}$ is an irrep of $\Sigma(216)$. Actually there are three eight-dimensional irreps:

$$\mathbf{8}^{(p)} = \mathbf{1}^{(p)} \otimes \mathbf{8}^{(0)} \quad (p = 0, 1, 2). \quad (133)$$

Since $3 \times 1^2 + 3 \times 2^2 + 3^2 + 3 \times 8^2 = 216$, we have found all irreps of $\Sigma(216)$ —see also [11]. Its character table is presented in table 13.

We still have to find 14 irreps of $\Sigma(216 \times 3)$ which are not irreps of $\Sigma(216)$. Multiplying the defining irrep by the one-dimensional irreps and taking complex conjugates we obtain six three-dimensional irreps:

$$\mathbf{3}^{(p)} = \mathbf{1}^{(p)} \otimes \mathbf{3}^{(0)}, \quad (\mathbf{3}^{(p)})^* = (\mathbf{1}^{(p)} \otimes \mathbf{3}^{(0)})^* \quad (p = 0, 1, 2). \quad (134)$$

We can also construct six six-dimensional irreps by

$$\mathbf{3}^{(0)} \otimes \mathbf{3}^{(0)} = (\mathbf{3}^{(0)})^* \oplus (\mathbf{6}^{(0)})^* \quad (135)$$

and

$$\mathbf{6}^{(p)} = \mathbf{1}^{(p)} \otimes \mathbf{6}^{(0)}, \quad (\mathbf{6}^{(p)})^* = (\mathbf{1}^{(p)} \otimes \mathbf{6}^{(0)})^* \quad (p = 0, 1, 2). \quad (136)$$

Finally, according to formula (25), there are two irreps left with dimensions d, d' fulfilling $d^2 + d'^2 = 162$. This equation has the unique solution $d = d' = 9$. An explicit construction of these nine-dimensional irreps is given by

$$\mathbf{9} = \mathbf{3}^{(0)} \otimes \mathbf{3}^{(a)} \quad \text{and} \quad \mathbf{9}^* = (\mathbf{3}^{(0)})^* \otimes \mathbf{3}^{(a)}. \quad (137)$$

The proof of the irreducibility of the irrep $\mathbf{9}$ is presented in appendix D. Thus we have completed the task of constructing all irreps of $\Sigma(216 \times 3)$. Its character table is divided into two parts, tables 14a and 14b, because it does not fit onto one page. The characters of the six, eight and nine-dimensional irreps are computed via

$$\chi_{\mathbf{6}^{(p)}} = \chi_{\mathbf{1}^{(p)}} \cdot \left[(\chi_{\mathbf{3}^{(0)}}^*)^2 - \chi_{\mathbf{3}^{(0)}} \right], \quad \chi_{\mathbf{8}^{(p)}} = \chi_{\mathbf{1}^{(p)}} \cdot (|\chi_{\mathbf{3}^{(0)}}|^2 - 1), \quad \chi_{\mathbf{9}} = \chi_{\mathbf{3}^{(0)}} \cdot \chi_{\mathbf{3}^{(a)}}. \quad (138)$$

6 Conclusions

In this paper we have performed a thorough discussion of the “exceptional” finite subgroups $\Sigma(36 \times 3)$, $\Sigma(72 \times 3)$ and $\Sigma(216 \times 3)$ of $SU(3)$ by means of the concept of principal series. These are maximal chains of ascending normal subgroups such that each member is a normal subgroup of all other groups higher up in the chain. Through their principal series the three groups under discussion have relationships which are useful for understanding their structures. For instance, all three principal series contain the sequence

$\Sigma(216 \times 3)$ (# C_k) $\text{ord}(C_k)$	C_1'' (1) 1	$\omega C_2''$ (1) 3	C_3'' (1) 3	C_4'' (24) 3	C_5'' (9) 2	C_6'' (9) 6	C_7'' (9) 6	C_8'' (54) 4	C_9'' (54) 12	C_{10}'' (54) 12	C_{11}'' (72) 3	C_{12}'' (72) 3
$\mathbf{1}^{(0)}$	1	1	1	1	1	1	1	1	1	1	1	1
$\mathbf{1}^{(1)}$	1	1	1	1	1	1	1	1	1	1	ω	ω^2
$\mathbf{1}^{(2)}$	1	1	1	1	1	1	1	1	1	1	ω^2	ω
$\mathbf{2}^{(0)}$	2	2	2	2	-2	-2	-2	0	0	0	-1	-1
$\mathbf{2}^{(1)}$	2	2	2	2	-2	-2	-2	0	0	0	$-\omega$	$-\omega^2$
$\mathbf{2}^{(2)}$	2	2	2	2	-2	-2	-2	0	0	0	$-\omega^2$	$-\omega$
$\mathbf{3}^{(a)}$	3	3	3	3	3	3	3	-1	-1	-1	0	0
$\mathbf{3}^{(0)}$	3	3ω	$3\omega^2$	0	-1	$-\omega$	$-\omega^2$	1	ω	ω^2	0	0
$\mathbf{3}^{(1)}$	3	3ω	$3\omega^2$	0	-1	$-\omega$	$-\omega^2$	1	ω	ω^2	0	0
$\mathbf{3}^{(2)}$	3	3ω	$3\omega^2$	0	-1	$-\omega$	$-\omega^2$	1	ω	ω^2	0	0
$(\mathbf{3}^{(0)})^*$	3	$3\omega^2$	3ω	0	-1	$-\omega^2$	$-\omega$	1	ω^2	ω	0	0
$(\mathbf{3}^{(1)})^*$	3	$3\omega^2$	3ω	0	-1	$-\omega^2$	$-\omega$	1	ω^2	ω	0	0
$(\mathbf{3}^{(2)})^*$	3	$3\omega^2$	3ω	0	-1	$-\omega^2$	$-\omega$	1	ω^2	ω	0	0
$\mathbf{6}^{(0)}$	6	6ω	$6\omega^2$	0	2	2ω	$2\omega^2$	0	0	0	0	0
$\mathbf{6}^{(1)}$	6	6ω	$6\omega^2$	0	2	2ω	$2\omega^2$	0	0	0	0	0
$\mathbf{6}^{(2)}$	6	6ω	$6\omega^2$	0	2	2ω	$2\omega^2$	0	0	0	0	0
$(\mathbf{6}^{(0)})^*$	6	$6\omega^2$	6ω	0	2	$2\omega^2$	2ω	0	0	0	0	0
$(\mathbf{6}^{(1)})^*$	6	$6\omega^2$	6ω	0	2	$2\omega^2$	2ω	0	0	0	0	0
$(\mathbf{6}^{(2)})^*$	6	$6\omega^2$	6ω	0	2	$2\omega^2$	2ω	0	0	0	0	0
$\mathbf{8}^{(0)}$	8	8	8	-1	0	0	0	0	0	0	-1	-1
$\mathbf{8}^{(1)}$	8	8	8	-1	0	0	0	0	0	0	$-\omega$	$-\omega^2$
$\mathbf{8}^{(2)}$	8	8	8	-1	0	0	0	0	0	0	$-\omega^2$	$-\omega$
$\mathbf{9}$	9	9ω	$9\omega^2$	0	-3	-3ω	$-3\omega^2$	-1	$-\omega$	$-\omega^2$	0	0
$\mathbf{9}^*$	9	$9\omega^2$	9ω	0	-3	$-3\omega^2$	-3ω	-1	$-\omega^2$	$-\omega$	0	0

Table 14a: Character table of $\Sigma(216 \times 3)$, part 1.

$\Sigma(216 \times 3)$ (# C_k) ord(C_k)	C''_{13} (12) 9	C''_{14} (12) 9	C''_{15} (12) 9	C''_{16} (12) 9	C''_{17} (12) 9	C''_{18} (12) 9	C''_{19} (36) 18	C''_{20} (36) 18	C''_{21} (36) 18	C''_{22} (36) 18	C''_{23} (36) 18	C''_{24} (36) 18
$\mathbf{1}^{(0)}$	1	1	1	1	1	1	1	1	1	1	1	1
$\mathbf{1}^{(1)}$	ω	ω	ω	ω^2	ω^2	ω^2	ω	ω	ω	ω^2	ω^2	ω^2
$\mathbf{1}^{(2)}$	ω^2	ω^2	ω^2	ω	ω	ω	ω^2	ω^2	ω^2	ω	ω	ω
$\mathbf{2}^{(0)}$	-1	-1	-1	-1	-1	-1	1	1	1	1	1	1
$\mathbf{2}^{(1)}$	$-\omega$	$-\omega$	$-\omega$	$-\omega^2$	$-\omega^2$	$-\omega^2$	ω	ω	ω	ω^2	ω^2	ω^2
$\mathbf{2}^{(2)}$	$-\omega^2$	$-\omega^2$	$-\omega^2$	$-\omega$	$-\omega$	$-\omega$	ω^2	ω^2	ω^2	ω	ω	ω
$\mathbf{3}^{(a)}$	0	0	0	0	0	0	0	0	0	0	0	0
$\mathbf{3}^{(0)}$	$\omega^2\sigma^*$	σ^*	$\omega\sigma^*$	σ	$\omega\sigma$	$\omega^2\sigma$	$-\omega\rho^*$	$-\omega^2\rho^*$	$-\rho^*$	$-\omega^2\rho$	$-\rho$	$-\omega\rho$
$\mathbf{3}^{(1)}$	σ^*	$\omega\sigma^*$	$\omega^2\sigma^*$	$\omega^2\sigma$	σ	$\omega\sigma$	$-\omega^2\rho^*$	$-\rho^*$	$-\omega\rho^*$	$-\omega\rho$	$-\omega^2\rho$	$-\rho$
$\mathbf{3}^{(2)}$	$\omega\sigma^*$	$\omega^2\sigma^*$	σ^*	$\omega\sigma$	$\omega^2\sigma$	σ	$-\rho^*$	$-\omega\rho^*$	$-\omega^2\rho^*$	$-\rho$	$-\omega\rho$	$-\omega^2\rho$
$(\mathbf{3}^{(0)})^*$	$\omega\sigma$	σ	$\omega^2\sigma$	σ^*	$\omega^2\sigma^*$	$\omega\sigma^*$	$-\omega^2\rho$	$-\omega\rho$	$-\rho$	$-\omega\rho^*$	$-\rho^*$	$-\omega^2\rho^*$
$(\mathbf{3}^{(1)})^*$	σ	$\omega^2\sigma$	$\omega\sigma$	$\omega\sigma^*$	σ^*	$\omega^2\sigma^*$	$-\omega\rho$	$-\rho$	$-\omega^2\rho$	$-\omega^2\rho^*$	$-\omega\rho^*$	$-\rho^*$
$(\mathbf{3}^{(2)})^*$	$\omega^2\sigma$	$\omega\sigma$	σ	$\omega^2\sigma^*$	$\omega\sigma^*$	σ^*	$-\rho$	$-\omega^2\rho$	$-\omega\rho$	$-\rho^*$	$-\omega^2\rho^*$	$-\omega\rho^*$
$\mathbf{6}^{(0)}$	$-\omega\sigma^*$	$-\omega^2\sigma^*$	$-\sigma^*$	$-\omega\sigma$	$-\omega^2\sigma$	$-\sigma$	$-\rho^*$	$-\omega\rho^*$	$-\omega^2\rho^*$	$-\rho$	$-\omega\rho$	$-\omega^2\rho$
$\mathbf{6}^{(1)}$	$-\omega^2\sigma^*$	$-\sigma^*$	$-\omega\sigma^*$	$-\sigma$	$-\omega\sigma$	$-\omega^2\sigma$	$-\omega\rho^*$	$-\omega^2\rho^*$	$-\rho^*$	$-\omega^2\rho$	$-\rho$	$-\omega\rho$
$\mathbf{6}^{(2)}$	$-\sigma^*$	$-\omega\sigma^*$	$-\omega^2\sigma^*$	$-\omega^2\sigma$	$-\sigma$	$-\omega\sigma$	$-\omega^2\rho^*$	$-\rho^*$	$-\omega\rho^*$	$-\omega\rho$	$-\omega^2\rho$	$-\rho$
$(\mathbf{6}^{(0)})^*$	$-\omega^2\sigma$	$-\omega\sigma$	$-\sigma$	$-\omega^2\sigma^*$	$-\omega\sigma^*$	$-\sigma^*$	$-\rho$	$-\omega^2\rho$	$-\omega\rho$	$-\rho^*$	$-\omega^2\rho^*$	$-\omega\rho^*$
$(\mathbf{6}^{(1)})^*$	$-\omega\sigma$	$-\sigma$	$-\omega^2\sigma$	$-\sigma^*$	$-\omega^2\sigma^*$	$-\omega\sigma^*$	$-\omega^2\rho$	$-\omega\rho$	$-\rho$	$-\omega\rho^*$	$-\rho^*$	$-\omega^2\rho^*$
$(\mathbf{6}^{(2)})^*$	$-\sigma$	$-\omega^2\sigma$	$-\omega\sigma$	$-\omega\sigma^*$	$-\sigma^*$	$-\omega^2\sigma^*$	$-\omega\rho$	$-\rho$	$-\omega^2\rho$	$-\omega^2\rho^*$	$-\omega\rho^*$	$-\rho^*$
$\mathbf{8}^{(0)}$	2	2	2	2	2	2	0	0	0	0	0	0
$\mathbf{8}^{(1)}$	2ω	2ω	2ω	$2\omega^2$	$2\omega^2$	$2\omega^2$	0	0	0	0	0	0
$\mathbf{8}^{(2)}$	$2\omega^2$	$2\omega^2$	$2\omega^2$	2ω	2ω	2ω	0	0	0	0	0	0
$\mathbf{9}$	0	0	0	0	0	0	0	0	0	0	0	0
$\mathbf{9}^*$	0	0	0	0	0	0	0	0	0	0	0	0

Table 14b: Character table of $\Sigma(216 \times 3)$, part 2, with $\rho \equiv e^{2\pi i/9}$, $\sigma \equiv \rho(1 + 2\omega)$.

$\Delta(27) \triangleleft \Delta(54)$. Using the principal series as a tool we have computed the conjugacy classes, irreps and character tables.

For finding the irreps the most useful property of a principal series (9) of a group G is that irreps of the factor groups G/G_k are also irreps of G itself. Most of the time the factor groups are relatively uncomplex and it is easy to find their irreps. Apart from very small abelian factor groups like \mathbb{Z}_2 , \mathbb{Z}_3 and $\mathbb{Z}_2 \times \mathbb{Z}_2$, among the factor groups occurring in the paper there are three interesting groups, two of which are widely used in model building, namely $\Sigma(72 \times 3)/\Delta(27) \cong Q_8$, $\Sigma(216 \times 3)/\Delta(27) \cong T'$ and $\Sigma(216 \times 3)/\Delta(54) \cong A_4$.

Since we have provided the character tables, in principle one can reduce any tensor product of irreps of the exceptional groups discussed in the paper. We have explicitly performed the reduction into irreps for tensor products of three-dimensional irreps [14]; this could be useful for model building. Particularly noteworthy are the very unusual Clebsch–Gordan coefficients occurring in $\mathbf{3}^{(0)} \otimes \mathbf{3}^{(0)}$ of $\Sigma(36 \times 3)$ —see [14] and appendix C of the present paper, and the occurrence of nine-dimensional irreps in the case of $\Sigma(216 \times 3)$.

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A Semidirect products of groups

Semidirect products are ubiquitous in the theory of finite groups. Therefore, we present in this appendix the definition and their most important properties.

Suppose we have two groups G and H and a homomorphism $\phi : G \rightarrow \text{Aut}(H)$ where $\text{Aut}(H)$ is the group of automorphisms of H , *i.e.* the group of isomorphisms $H \rightarrow H$. Let us dwell a bit on ϕ before we go on to the definition of the semidirect product $H \rtimes G$. We denote the unit elements of H , G and $\text{Aut}(H)$ by e , e' and id , respectively. Since ϕ is a homomorphism, the relations $\phi(g_1 g_2) = \phi(g_1) \phi(g_2)$ and $\phi(e') = \text{id}$ hold. Moreover, $\phi(g)(h_1 h_2) = (\phi(g)h_1)(\phi(g)h_2)$ since $\phi(g)$ is an automorphism on H .

A.1 Definition. A semidirect product $H \rtimes_\phi G$ of two groups H and G is defined as the set $H \times G$ with the multiplication law

$$(h_1, g_1)(h_2, g_2) = (h_1 \phi(g_1)h_2, g_1 g_2), \quad (\text{A.1})$$

where ϕ is a homomorphism $\phi : G \rightarrow \text{Aut}(H)$.

Note that the definition depends on the homomorphism ϕ ; via ϕ the group G acts on H . For simplicity of notation we will drop the index ϕ at the symbol \rtimes in the following.

The relevant property of $H \rtimes G$ for group theory is the following.

A.2 Theorem. The semidirect product $H \rtimes G$ laid down in the definition A.1 is a group.

To establish the group property we note that (e, e') is the unit element of $H \rtimes G$ and that

$$(h, g)^{-1} = (\phi(g^{-1})h^{-1}, g^{-1}). \quad (\text{A.2})$$

Only the verification of associativity of the multiplication law is a bit lengthier and we leave this as an exercise to the reader.

The multiplication law (A.1) is rather abstract. However, the following discussion attempts to make it more transparent. First we note that the relations

$$(h_1, e')(h_2, e') = (h_1 h_2, e') \quad \text{and} \quad (e, g_1)(e, g_2) = (e, g_1 g_2) \quad (\text{A.3})$$

hold. This means that both $H \times \{e'\}$ and $\{e\} \times G$ are subgroups of $H \rtimes G$. Moreover, due to

$$(h, g)(h', e')(h, g)^{-1} = (h(\phi(g)h')h^{-1}, e'), \quad (\text{A.4})$$

H is a normal subgroup. In addition, any pair (h, g) can uniquely be decomposed into

$$(h, g) = (h, e')(e, g) = (e, g)(\bar{h}, e') \quad \text{with} \quad \bar{h} = \phi(g^{-1})h. \quad (\text{A.5})$$

This allows to write the multiplication law (A.1) as

$$(h_1, g_1)(h_2, g_2) = (h_1, e')(e, g_1)(h_2, e')(e, g_1)^{-1}(e, g_1)(e, g_2) \quad (\text{A.6})$$

with $(h_1, e')(e, g_1)(h_2, e')(e, g_1)^{-1} \in H \times \{e'\}$ and $(e, g_1)(e, g_2) \in \{e\} \times G$.

The usefulness and ubiquity of semidirect products has its roots in the following theorem.

A.3 Theorem. If S is a group with a normal subgroup H and a subgroup G such that

1. $H \cap G = \{e\}$, where e is the unit element of S , and
2. every element $s \in S$ can be written as $s = hg$ with $h \in H$ and $g \in G$,

then the decomposition $s = hg$ is unique, $S/H \cong G$ and, via $s = hg \rightarrow (h, g)$, the group S is isomorphic to $H \rtimes_\phi G$ with $\phi(g)h = ghg^{-1}$.

Proof: First we show the uniqueness of the decomposition of $s \in S$. Let us assume that $s = hg = h'g'$ with $h, h' \in H$ and $g, g' \in G$. This assumption leads to the relation $h'^{-1}h = g'g^{-1}$ with the element on the left-hand side being in H and the element on the right-hand side being in G . Since the intersection of H and G consists of the unit element only, we find $h = h'$ and $g = g'$, *i.e.* uniqueness of the decomposition. The isomorphism between S/H and G is given by $Hg \leftrightarrow g$. Now we assume that we have elements $s_1 = h_1g_1$ and $s_2 = h_2g_2$ of S . Then the isomorphism between S and $H \rtimes G$ follows readily from

$$s_1 s_2 = h_1 g_1 h_2 g_2 = h_1 (g_1 h_2 g_1^{-1}) g_1 g_2 \rightarrow (h_1 (g_1 h_2 g_1^{-1}), g_1 g_2) = (h_1, g_1)(h_2, g_2). \quad (\text{A.7})$$

Q.E.D.

We stress that in equation (A.7) the mapping $\phi(g)h = ghg^{-1}$ of G into $\text{Aut}(H)$ comes about simply by the group multiplication in S and reordering of the factors in the product $s_1 s_2$ in order to recover the decomposition of theorem A.3. This is completely analogous to the procedure leading to equation (A.6).

B $\Delta(27)$, $\Delta(54)$ and tensor products of their three-dimensional irreps

Introductory remarks: Before we discuss tensor products of $\Delta(27)$ and $\Delta(54)$, some introductory remarks concerning $SU(3)$ subgroups are at order. Such a subgroup G can also be conceived as a three-dimensional representation which we denote generically by $\underline{3}$. Moreover, with $\underline{3}$ we automatically also have the complex conjugate representation $\underline{3}^*$. Therefore, the $SU(3)$ relations for the tensor products

$$\underline{3} \otimes \underline{3}^* = \underline{1} \oplus \underline{8}, \quad \underline{3} \otimes \underline{3} = \underline{3}^* \oplus \underline{6} \quad (\text{B.1})$$

hold. If we denote the Cartesian unit vectors of \mathbb{C}^3 by e_j ($j = 1, 2, 3$), the $\underline{1}$ is the trivial irrep acting on the basis vector

$$\frac{1}{\sqrt{3}}(e_1 \otimes e_1 + e_2 \otimes e_2 + e_3 \otimes e_3), \quad (\text{B.2})$$

while the $\underline{3}^*$ resides in the subspace spanned by the vectors

$$a_j = \frac{1}{\sqrt{2}} \epsilon_{jkl} e_k \otimes e_l, \quad (\text{B.3})$$

where the ϵ_{jkl} are the components of the totally antisymmetric ϵ -tensor. Let us assume that $\underline{3}$ is irreducible. Then we make the following useful observations:

- For $G \subset SU(3)$ the $\underline{6}$ and the $\underline{8}$ may be irreducible or not.
- If G contains the center of $SU(3)$, then the $\underline{8}$ represents the center trivially.

A thorough discussion of tensor products of three-dimensional irreps of finite subgroups of $SU(3)$ is found in [14].

The tensor product $\underline{3} \otimes \underline{3}^*$: It is convenient to exploit the fact that this tensor product decays into nine one-dimensional irreps under $\Delta(27)$, *i.e.* one can choose a common basis of eigenvectors of $C \otimes C^*$ and $E \otimes E$:

$$\begin{aligned} b_{00} &= \frac{1}{\sqrt{3}}(e_1 \otimes e_1 + e_2 \otimes e_2 + e_3 \otimes e_3), \\ b_{01} &= \frac{1}{\sqrt{3}}(e_1 \otimes e_1 + \omega e_2 \otimes e_2 + \omega^2 e_3 \otimes e_3), \\ b_{02} &= \frac{1}{\sqrt{3}}(e_1 \otimes e_1 + \omega^2 e_2 \otimes e_2 + \omega e_3 \otimes e_3), \\ b_{10} &= \frac{1}{\sqrt{3}}(e_2 \otimes e_1 + e_3 \otimes e_2 + e_1 \otimes e_3), \\ b_{11} &= \frac{1}{\sqrt{3}}(e_2 \otimes e_1 + \omega e_3 \otimes e_2 + \omega^2 e_1 \otimes e_3), \\ b_{12} &= \frac{1}{\sqrt{3}}(e_1 \otimes e_3 + \omega^2 e_2 \otimes e_1 + \omega e_3 \otimes e_2), \\ b_{20} &= \frac{1}{\sqrt{3}}(e_1 \otimes e_2 + e_2 \otimes e_3 + e_3 \otimes e_1), \\ b_{21} &= \frac{1}{\sqrt{3}}(e_1 \otimes e_2 + \omega e_2 \otimes e_3 + \omega^2 e_3 \otimes e_1), \\ b_{22} &= \frac{1}{\sqrt{3}}(e_3 \otimes e_1 + \omega^2 e_1 \otimes e_2 + \omega e_2 \otimes e_3) \end{aligned} \quad (\text{B.4})$$

The eigenvalues under the action of C and E are given by

$$(C \otimes C^*) b_{pq} = \omega^p b_{pq} \quad \text{and} \quad (E \otimes E) b_{pq} = \omega^q b_{pq}, \quad (\text{B.5})$$

respectively. Consequently, with equation (24) we arrive at the result

$$\Delta(27) : \quad \mathbf{3} \otimes \mathbf{3}^* = \bigoplus_{p,q=0}^2 \mathbf{1}^{(p,q)}, \quad (\text{B.6})$$

which nicely illustrates the decay of the $\underline{8}$ of equation (B.1) into irreps, in this case into eight one-dimensional irreps.

Moving to $\Delta(54)$, we have to apply V^2 to the basis vectors (B.4) which has the effect that, apart from b_{00} , these basis vectors are grouped in two to form the two-dimensional irreps:

$$\mathbf{2} : \{b_{01}, b_{02}\}, \quad \mathbf{2}' : \{b_{10}, b_{20}\}, \quad \mathbf{2}'' : \{b_{11}, b_{22}\}, \quad \mathbf{2}''' : \{b_{12}, b_{21}\}. \quad (\text{B.7})$$

The labels for the irreps refer to equations (47) and (48). Thus, we have just derived

$$\Delta(54) : \quad \mathbf{3} \otimes \mathbf{3}^* = \mathbf{1} \oplus \mathbf{2} \oplus \mathbf{2}' \oplus \mathbf{2}'' \oplus \mathbf{2}'''. \quad (\text{B.8})$$

The tensor product $\mathbf{3} \otimes \mathbf{3}$: We complete the basis vectors (B.3) to a basis of $\mathbb{C}^3 \otimes \mathbb{C}^3$ by adding the sets of basis vectors

$$\{e_1 \otimes e_1, e_2 \otimes e_2, e_3 \otimes e_3\} \quad (\text{B.9})$$

and

$$\left\{ \frac{1}{\sqrt{2}} (e_2 \otimes e_3 + e_3 \otimes e_2), \frac{1}{\sqrt{2}} (e_3 \otimes e_1 + e_1 \otimes e_3), \frac{1}{\sqrt{2}} (e_1 \otimes e_2 + e_2 \otimes e_1) \right\}. \quad (\text{B.10})$$

On both sets, the action of tensor products of the group generators is given by

$$C \otimes C \rightarrow C^*, \quad E \otimes E \rightarrow E, \quad V^2 \otimes V^2 \rightarrow -V^2. \quad (\text{B.11})$$

Therefore, we obtain

$$\Delta(27) : \quad \mathbf{3} \otimes \mathbf{3} = \mathbf{3}^* \oplus \mathbf{3}^* \oplus \mathbf{3}^* \quad (\text{B.12})$$

and

$$\Delta(54) : \quad \mathbf{3} \otimes \mathbf{3} = \mathbf{3}^* \oplus (\mathbf{3}')^* \oplus (\mathbf{3}')^* \quad (\text{B.13})$$

For the irreps and the character table of $\Delta(27)$ see section 2.1, for those of $\Delta(54)$ see section 2.2. Of course, equations (B.6), (B.8), (B.12) and (B.13) could have also been derived from the corresponding character tables.

C Tensor products of of three-dimensional irreps of $\Sigma(36 \times 3)$

The four-dimensional irreps of $\Sigma(36 \times 3)$: We denote the defining irrep (50) by $\mathbf{3}^{(0)}$. We know already from section 3.3 that the four-dimensional irreps are irreps of $\Sigma(36)$ which has a trivial center. This suggests, according to the introduction of appendix B, to consider $\mathbf{3}^{(0)} \otimes (\mathbf{3}^{(0)})^*$ in search of these irreps. We again use the basis (B.4) and the fact that it is a basis of eigenvectors of C and E . It remains to compute the action of $V \otimes V^*$ on b_{pq} . The result of the tedious computation can graphically be presented in the following way:

$$V \otimes V^* : \begin{array}{ccccc} b_{01} & \rightarrow & b_{20} & b_{11} & \rightarrow & b_{21} \\ \uparrow & & \downarrow & \uparrow & & \downarrow \\ b_{10} & \leftarrow & b_{02} & b_{12} & \leftarrow & b_{22} \end{array} \quad (\text{C.1})$$

It is easy to check that the matrices of generators in the four-dimensional irreps of equation (72) and (73) are obtained by the following ordering of the basis elements:

$$\begin{aligned} \mathbf{4} : & \{b_{01}, b_{10}, b_{02}, b_{20}\}, \\ \mathbf{4}' : & \{b_{11}, b_{12}, b_{22}, b_{21}\}. \end{aligned} \quad (\text{C.2})$$

Thus the decomposition of $\mathbf{3}^{(0)} \otimes (\mathbf{3}^{(0)})^*$ is given by

$$\mathbf{3}^{(0)} \otimes (\mathbf{3}^{(0)})^* = \mathbf{1}^{(0)} \oplus \mathbf{4} \oplus \mathbf{4}'. \quad (\text{C.3})$$

Taking the definition of the irreps $\mathbf{3}^{(p)}$ from equation (75), the relation (C.3) can readily be generalized to

$$\mathbf{3}^{(p)} \otimes (\mathbf{3}^{(p')})^* = \mathbf{1}^{(p-p')} \oplus \mathbf{4} \oplus \mathbf{4}', \quad (\text{C.4})$$

where $p-p'$ has to be taken modulo 4. In this equation we have used that $\mathbf{1}^{(p)} \otimes \mathbf{4} \cong \mathbf{4}$ and $\mathbf{1}^{(p)} \otimes \mathbf{4}' \cong \mathbf{4}'$, which can be read off from the character table because $\chi_{\alpha k} \cdot \chi_{\beta k} = \chi_{\beta k}$ for all conjugacy classes C_k whenever α denotes a one-dimensional and β a four-dimensional irrep.

The tensor product $\mathbf{3}^{(0)} \otimes \mathbf{3}^{(0)}$: According to equation (B.1), this tensor product contains a $(\mathbf{3}^{(0)})^*$ which resides in the subspace spanned by the basis vectors (B.3). Since $\Sigma(36 \times 3)$ has no six-dimensional irrep, the symmetric part of the tensor product must decay into invariant subspaces. To verify this statement, we choose an orthogonal system of vectors

$$\begin{aligned} \bar{f}_1 &= e_1 \otimes e_1 + \zeta(e_2 \otimes e_3 + e_3 \otimes e_2), \\ \bar{f}_2 &= e_2 \otimes e_2 + \zeta(e_3 \otimes e_1 + e_1 \otimes e_3), \\ \bar{f}_3 &= e_3 \otimes e_3 + \zeta(e_1 \otimes e_2 + e_2 \otimes e_1), \end{aligned} \quad (\text{C.5})$$

where ζ is a parameter. Note that the action of C and E on the \bar{f}_j can be characterized as

$$C \otimes C : \bar{f}_j \rightarrow \omega^{2(j-1)} \bar{f}_j \quad (j = 1, 2, 3), \quad E \otimes E : \bar{f}_1 \rightarrow \bar{f}_3 \rightarrow \bar{f}_2 \rightarrow \bar{f}_1, \quad (\text{C.6})$$

which shows that, on the basis vectors \bar{f}_j , C and E are represented by C^* and E , respectively. Next we consider the action of V on the \bar{f}_j , in order to investigate if for specific values of ζ the \bar{f}_j form the basis of an invariant space:

$$(V \otimes V) \bar{f}_j = M_{kj} \bar{f}_k. \quad (\text{C.7})$$

Writing

$$\bar{f}_j = \Gamma_{kl}^j e_k \otimes e_l \quad \text{with} \quad \Gamma^1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & \zeta \\ 0 & \zeta & 0 \end{pmatrix} \quad \text{etc.}, \quad (\text{C.8})$$

equation (C.7) reads

$$V \Gamma^j V^T = M_{kj} \Gamma^k. \quad (\text{C.9})$$

Evaluating this equation and using $V^T = V$, we find

$$V \Gamma^1 V = -\frac{1}{3} \begin{pmatrix} 1+2\zeta & 1-\zeta & 1-\zeta \\ 1-\zeta & 1+2\zeta & 1-\zeta \\ 1-\zeta & 1-\zeta & 1+2\zeta \end{pmatrix} = M_{k1} \Gamma^k, \quad (\text{C.10a})$$

$$V \Gamma^2 V = -\frac{1}{3} \begin{pmatrix} 1+2\zeta & (1-\zeta)\omega & (1-\zeta)\omega^2 \\ (1-\zeta)\omega & (1+2\zeta)\omega^2 & 1-\zeta \\ (1-\zeta)\omega^2 & 1-\zeta & (1+2\zeta)\omega \end{pmatrix} = M_{k2} \Gamma^k, \quad (\text{C.10b})$$

$$V \Gamma^3 V = -\frac{1}{3} \begin{pmatrix} 1+2\zeta & (1-\zeta)\omega^2 & (1-\zeta)\omega \\ (1-\zeta)\omega^2 & (1+2\zeta)\omega & 1-\zeta \\ (1-\zeta)\omega & 1-\zeta & (1+2\zeta)\omega^2 \end{pmatrix} = M_{k3} \Gamma^k. \quad (\text{C.10c})$$

It is straightforward to check that these equations lead to the consistent solution

$$M = -\frac{1+2\zeta}{3} \begin{pmatrix} 1 & 1 & 1 \\ 1 & \omega^2 & \omega \\ 1 & \omega & \omega^2 \end{pmatrix}, \quad (\text{C.11})$$

provided $1+2\zeta = (1-\zeta)/\zeta$. This leads to the quadratic equation

$$2\zeta^2 + 2\zeta - 1 = 0, \quad (\text{C.12})$$

with the solutions

$$\zeta_{\pm} = \frac{-1 \pm \sqrt{3}}{2}. \quad (\text{C.13})$$

Therefore, our result is

$$-\frac{1+2\zeta_{\pm}}{3} = \mp \frac{1}{\sqrt{3}} \Rightarrow M = (\mp iV)^*. \quad (\text{C.14})$$

Comparing with equation (75), we have derived

$$\mathbf{3}^{(0)} \otimes \mathbf{3}^{(0)} = (\mathbf{3}^{(0)})^* \oplus (\mathbf{3}^{(1)})^* \oplus (\mathbf{3}^{(3)})^*, \quad (\text{C.15})$$

where the upper signs in equations (C.13) and (C.14) refer to the $(\mathbf{3}^{(3)})^*$ and the lower signs to the $(\mathbf{3}^{(1)})^*$. The corresponding basis vectors normalized to one are given by

$$f_j^{(\pm)} = \frac{\tau_{\pm}}{\sqrt{12}} e_j \otimes e_j \pm \frac{1}{\tau_{\pm}} (e_k \otimes e_l + e_l \otimes e_k) \quad \text{with } j \neq k \neq l \neq j \text{ and } \tau_{\pm} = \sqrt{2(3 \pm \sqrt{3})}, \quad (\text{C.16})$$

for $(\mathbf{3}^{(3)})^*$ and $(\mathbf{3}^{(1)})^*$, respectively. It is easy to check that these vectors, together with the a_j of equation (B.3), form an orthonormal basis for $\mathbf{3}^{(0)} \otimes \mathbf{3}^{(0)}$.

Equation (C.15) can be generalized to

$$\mathbf{3}^{(p)} \otimes \mathbf{3}^{(p')} = (\mathbf{3}^{(p_1)})^* \oplus (\mathbf{3}^{(p_2)})^* \oplus (\mathbf{3}^{(p_3)})^* \quad (\text{C.17})$$

with

$$p_1 = (-p - p') \bmod 4, \quad p_2 = (-p - p' + 1) \bmod 4, \quad p_3 = (-p - p' + 3) \bmod 4. \quad (\text{C.18})$$

D The nine-dimensional irreps of $\Sigma(216 \times 3)$

For the notation concerning the irreps of $\Sigma(216 \times 3)$ consult section 5.3.

D.1 Theorem. The tensor product $\mathbf{3}^{(0)} \otimes \mathbf{3}^{(a)}$ establishes a nine-dimensional irrep of $\Sigma(216 \times 3)$.

Proof: In order to proof this theorem we show that, for any non-zero vector

$$x = \sum_{i,j=1}^3 c_{ij} e_i \otimes e_j \in \mathbb{C}^3 \otimes \mathbb{C}^3, \quad (\text{C.1})$$

by application of the representation operators of $\mathbf{3}^{(0)} \otimes \mathbf{3}^{(a)}$ to x we obtain a set of vectors which spans the whole space. The e_i are the Cartesian basis vectors. First we observe that the elements $C, E, V, D \in \Sigma(216 \times 3)$ are represented by

$$C \rightarrow C \otimes \mathbb{1}, \quad E \rightarrow E \otimes \mathbb{1}, \quad V \rightarrow V \otimes R_a, \quad D \rightarrow D \otimes (R_a R_b), \quad (\text{C.2})$$

respectively, in $\mathbf{3}^{(0)} \otimes \mathbf{3}^{(a)}$. The matrices R_a, R_b are given in equation (120). At least one of the c_{ij} must be non-zero. Without loss of generality we can assume that there is an index k such that $c_{k1} \neq 0$. If this is not the case, by application of D we can always achieve this. In the next step we consider the vector

$$y \equiv [(\mathbb{1} + E + E^2) \otimes \mathbb{1}] x = \sum_{j=1}^3 d_j u \otimes e_j \quad \text{with} \quad d_j = \sum_{i=1}^3 c_{ij}, \quad u = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}. \quad (\text{C.3})$$

Again, without loss of generality, we can assume $d_1 \neq 0$. The reason is that we can apply the operator C or C^2 to x before we perform the operation (C.3). This would change $d_1 = \sum_{i=1}^3 c_{i1}$ to

$$d_1 = \sum_{i=1}^3 \omega^{(i-1)} c_{i1} \quad \text{or} \quad d_1 = \sum_{i=1}^3 \omega^{2(i-1)} c_{i1}. \quad (\text{C.4})$$

By assumption, not all three versions of d_1 can be zero at the same time because otherwise $c_{11} = c_{21} = c_{31} = 0$ in contradiction to one c_{k1} being non-zero.

Now we continue with the vector y and $d_1 \neq 0$. We note that $Vu = -i\sqrt{3}e_1$. This allows to construct the vector

$$z_1 = \frac{i}{\sqrt{3}}(V \otimes R_a)y = d_1 e_1 \otimes e_1 - d_2 e_1 \otimes e_2 - d_3 e_1 \otimes e_3. \quad (\text{C.5})$$

Furthermore, application of

$$\frac{1}{3}(\mathbb{1} + C + C^2) \otimes \mathbb{1} \quad (\text{C.6})$$

to y generates the vector

$$z_2 = \sum_{j=1}^3 d_j e_1 \otimes e_j. \quad (\text{C.7})$$

Therefore, we have shown that, starting with x , the vector $e_1 \otimes e_1 = (z_1 + z_2)/(2d_1)$ is necessarily in the representation space of $\mathbf{3}^{(0)} \otimes \mathbf{3}^{(a)}$. Then, repeated application of D and E to $e_1 \otimes e_1$ generates a basis of $\mathbb{C}^3 \otimes \mathbb{C}^3$. Q.E.D.

One can ask the question how many inequivalent nine-dimensional irreps exist. Because of

$$\mathbf{3}^{(p)} \otimes \mathbf{3}^{(a)} = (\mathbf{1}^{(p)} \otimes \mathbf{3}^{(0)}) \otimes \mathbf{3}^{(a)} \cong \mathbf{3}^{(0)} \otimes (\mathbf{1}^{(p)} \otimes \mathbf{3}^{(a)}) \cong \mathbf{3}^{(0)} \otimes \mathbf{3}^{(a)} \quad (\text{C.8})$$

there are only two, namely $\mathbf{9} \equiv \mathbf{3}^{(0)} \otimes \mathbf{3}^{(a)}$ and $\mathbf{9}^* \equiv (\mathbf{3}^{(0)})^* \otimes \mathbf{3}^{(a)}$.

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